

Suggested Problems:

- pgs 103-106-85 number 3a-c, 5 (no technology),7,13,17,19
- pgs 115-118 number 1,19

FUNCTIONS OF SEVERAL VARIABLES.

2D: If $f(x, y)$ is a function of two variables, then $f(x, y) = \text{const}$ is curve in the plane. For example, $f(x, y) = 4x^2 + 3y^2 = 1$ is an ellipse.

3D: If $f(x, y, z)$ is a function of three variables and c is a constant then $f(x, y, z) = c$ is a surface in space. For example, $f(x, y, z) = 4x^2 + 3y^2 + z^2 = 1$ is an ellipsoid.

PARTIAL DERIVATIVE. If $f(x, y, z)$ is a function of three variables, then $\frac{\partial}{\partial x} f(x, y, z)$ is defined as the derivative of the function $g(x) = f(x, y, z)$ where y and z are fixed. The other derivatives with respect to y and z are defined similarly.

REMARK. The partial derivatives measure the rate of change of the function in the x, y, z directions.

NOTATION. One also writes $f_x(x, y, z) = \frac{\partial}{\partial x} f(x, y, z)$ etc.

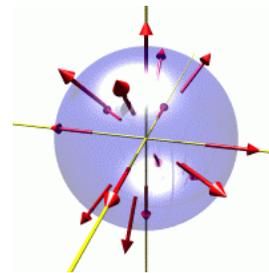
GRADIENT. If $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = \left(\frac{\partial}{\partial x} f(x, y, z), \frac{\partial}{\partial y} f(x, y, z), \frac{\partial}{\partial z} f(x, y, z) \right)$$

is called the **gradient** of f . The symbol ∇ is called **Nabla**.

NORMAL. The gradient $\nabla f(x, y)$ is orthogonal to the level curve $f(x, y) = c$. The gradient $\nabla f(x, y, z)$ is normal to the level surface $f(x, y, z)$.

It is enough to understand this for a plane because if we look close at a point P on a level surface, it looks as if the level surface is a plane. If $f(x, y, z) = ax + by + cz = d$ then $n = \nabla f(x, y, z) = (a, b, c)$ and this is normal to the plane because if two points $P_i = (x_i, y_i, z_i)$ are in the plane, then $ax_1 + by_1 + cz_1 = d$, and $ax_2 + by_2 + cz_2 = d$. Subtracting these two equations means that n is orthogonal to $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$.



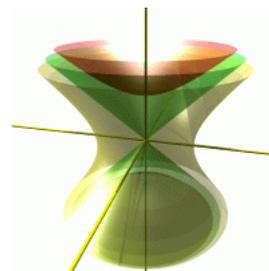
EXAMPLE. The gradient of $f(x, y, z) = x^2 + y^2 - z^2$ at a point (x, y, z) is $(2x, 2y, -2z)$.

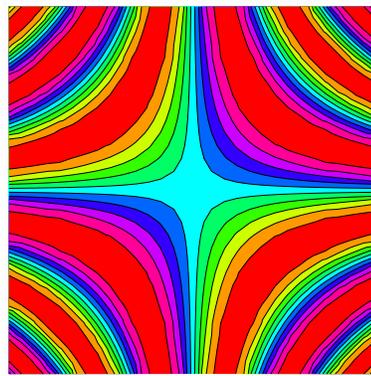
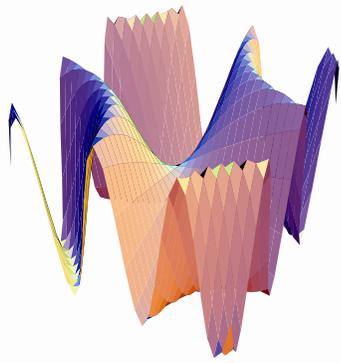
DIRECTION. The gradient points into the direction, for which f increases: We will later derive the formula $d/ds f(x + s\nabla f) = |\nabla f|^2$ which shows that the length of the gradient measures how f changes orthogonal to the surface. Take for example $f(x, y, z) = x^2 + y^2 + z^2$, where $\nabla f(x, y, z) = (2x, 2y, 2z)$. Then f increases if we move in the direction ∇f .

CONTOURS. If $f(x, y)$ is a function of two variables, then $f(x, y) = c$ are called a **contour line** of f . If $f(x, y, z)$ is a function of three variables, then $f(x, y, z) = c$ is called a **contour surfaces** of f .

EXAMPLE. Let $f(x, y) = x^2 - y^2$. Determine the contour lines to $f(x, y) = 0$, $f(x, y) = 1$, $f(x, y) = -1$. The set $x^2 - y^2 = 0$ is the union of the sets $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with their tips at $(-1, 0)$ and $(1, 0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their tips at $(0, \pm 1)$.

EXAMPLE. Let $f(x, y, z) = x^2 + y^2 - z^2$. Determine the contour surfaces $f(x, y, z) = 0$, $f(x, y, z) = 1$, $f(x, y, z) = -1$. The set $x^2 + y^2 - z^2 = 0$ is a **cone** rotational symmetric around the z -axes. The set $x^2 + y^2 - z^2 = 1$ is a **one-sheeted hyperboloid**, the set $x^2 + y^2 - z^2 = -1$ is a **two-sheeted hyperboloid**. (How to see that it is two-sheeted: the intersection with $z = c$ is empty for $-1 \leq z \leq 1$.)





TOPOGRAPHY. Topographical maps often show the curves of equal height. With this information it is usually already possible to have a good picture of the situation.



POTENTIAL. If the force $F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$ is the gradient of a function $U(x, y, z)$, then U is called a **potential**. In other words, then $f(x, y, z) = U_x(x, y, z)$, $g(x, y, z) = U_y(x, y, z)$ and $h(x, y, z) = U_z(x, y, z)$.

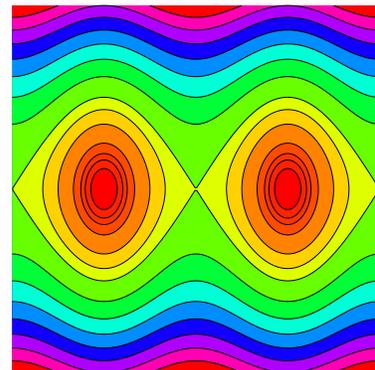
EXAMPLE. If $U(x, y, z) = 1/|x|$, then $\nabla U(x, y, z) = -x/|x|^3$. U is the **Coulomb potential** and ∇U is the **Coulomb force**.

EXAMPLES. $F(x, y) = (2xy^2, 2x^2y)$ has the potential $U(x, y) = x^2y^2$. $F(x, y) = (y, 0)$ has no potential (we will see later, why).

POTENTIAL ENERGY. The potential U of a force is also called the **potential energy**. If $r''(t) = F$ and $F = \nabla U$, then the **energy** $E(t) = m \frac{r'(t)^2}{2} - U(r(t))$ is conserved:

$$\frac{d}{dt}E(t) = r'(t) \cdot r''(t) - \nabla U(r(t)) \cdot r'(t) = r'(t) \cdot (r''(t) - \nabla U(r(t))) = 0.$$

EXAMPLE. $U(x) = g(1 - \cos(x))$ is the potential of a **pendulum**. The motion of the pendulum can be observed in the (x, v) -plane. This plane is called the **phase space**. If $E(x, v) = v^2/2 - U(x)$ is the energy of the pendulum, then $(x(t), y(t))$ stays on the level curves of the pendulum. The fact that the energy surface $E = c$ is a curve, restricts the dynamics to curves and forbids for example chaotic behavior.



For a **double pendulum**, where we have two angles and two angular velocities, the phase space is 4 dimensional. Energy is conserved and the motion happens on "three-dimensional contour sets". The double pendulum is known to behave chaotically.