Midterm: The first mideterm takes place today in one week, on October 10 at 7-9pm in Science Center C or D (either room). The topics are Chapter 1 + Appendix A. A study guide + 2 Practice Exams are on the web.

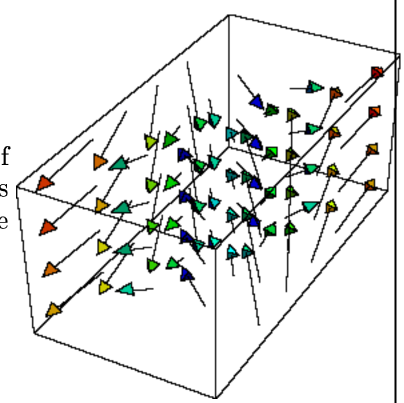
Homework:

- pgs 103-106 number 2a-c,6,12
- pgs 115-118 number 4,12,16,18
- pgs 254-255 number 2,6
- The function $u(x,y) = \cos(x+y)$ obeys the partial differential equation $u_x - u_y = 0$, where $u_x(x,y), u_y(x,y)$ are the partial derivatives with respect to x and y. Find two other non-constant functions, neither multiples of $\cos(x+y)$ that obey this equation.

Suggested Problems:

- pgs 115-118 number 11,13.
- pgs 254-255 number 1 (no technology) 3 (no technology) and 5
- Suppose that a particle is moved from the origin to the point $(1,1,1)$ along the line segment between them in the presence of the force $F = (xy, y^2 - z, 3y)$. How much work is done?

VECTOR FIELDS. If we attach a vector $F(x,y,z)$ at each point (x,y,z) of space, we obtain a **vector field**. A vector field is given by three functions $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$. Similarly, a vector field in the plane is given by $F(x,y) = (F_1(x,y), F_2(x,y))$.

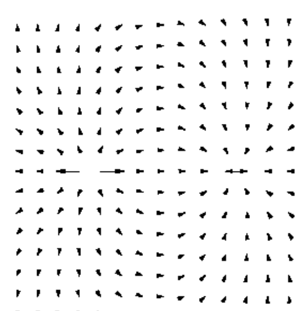


EXAMPLES.
1) $F(x,y) = (y,-x)$ is a planar vector field which you see in a picture on the right.

2) $F(x,y) = (x-1,y)/((x-1)^2+y^2)^{3/2} - (x+1,y)/((x+1)^2+y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipol field**. See picture.

3) **Gradient field**. If $f(x,y,z)$ is a function of three variables, then $\nabla f(x,y,z)$ is called the **gradient field**. For example, $(2x, 2y, -2z)$ is the vector field which is orthogonal to hyperboloids.

4) If $H(x,y)$ is a function of two variables, then $(H_y(x,y), -H_x(x,y))$ is called a **Hamiltonian vector field**. Example: Harmonic Oscillator $H(x,y) = x^2 + y^2$. $(H_y(x,y), H_x(x,y)) = (y,-x)$ is the vector field in 1).



WHERE DO VECTOR FIELDS OCCUR?
Vector fields appear often in physics.

Electric fields, magnetic fields, velocity fields, wind directions, pressure gradients, gravitational fields, force fields, etc.

this is not the case.

LINE INTEGRALS. If $F(x,y,z)$ is a vector field and $\gamma : t \mapsto r(t)$ is a curve, then $\int_a^b F(r(t)) \cdot r'(t)\ dt$ is called a **line integral** along the curve. The short-hand notation $\int_\gamma F\ \cdot ds$ is also used. In the book, where curves are sometimes written as $X(t) = (x(t), y(t), z(t))$, the notation $\int_\gamma F\ \cdot dX$ appears

ABOUT LINE INTEGRALS. We have seen that the existence of several variables allows differentiations with respect to each variables. What about integration in higher dimensions? While in one dimension, we could integrate from a to b, in the plane or in space, there are many different possibilities to integrate from a point P to a point Q.

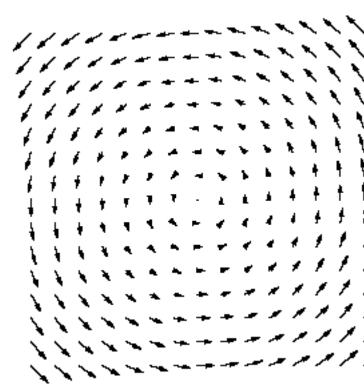
EXAMPLE: **Work**. If $F(x,y,z)$ is a force field, then the line integral $\int_a^b F(r(t)).r'(t)\ dt$ is called **work**.

EXAMPLE: **Electric potential**. If $E(x,y,z)$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r'(t)\ dt$ is called **electic potential**.

EXAMPLE: **Gradient field**. If $F(x,y,z) = \nabla U(x,y,z)$ is a gradient field, then $\int_a^b F(r(t)) \cdot r'(t)\ dt = U(r(b)) - U(r(a))$. The gradient field has physical relevance. For example. If $U(x,y,z)$ is the pressure distribution in the athmosphere, then $\nabla U(x,y,z)$ is the pressure gradient typically giving the wind directions.

CLOSED CURVES. We see from the last example that the line integral along a closed curve is zero if the vector field is a gradient field. The work done along a closed path is zero. This is a form of energy conservation.

EXAMPLE. Let $\gamma : t \mapsto r(t) = (\cos(t), \sin(t))$ be a circle parametrized by $t \in [0, 2\pi]$ and let $F(x,y) = (-y, x)$. Calculate the line integral $I = \int_\gamma F(r) \cdot dr$.



ANSWER: We have

$$
\begin{aligned}
I &= \int_0^{2\pi} F(r(t)) \cdot r'(t)\ dt \\
&= \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t))\ dt \\
&= \int_0^{2\pi} \cos^2(t) + \sin^2(t)\ dt = 2\pi\ .
\end{aligned}
$$

GENERALISATION OF THE FUNDAMENTAL THEOREM: (see later)

$$\int_a^b \nabla U(r(t)) \cdot r'(t)\ dt = U(r(b)) - U(r(a))$$

is a generalisation of the fundmantal theorem of calculus. It is a generalisation because for $r(t) = t$ in one dimension, we have $\int_a^b U_x(x)\ dx = U(b) - U(a)$. The proof above needs a multi-dimensional version $d/dt\, U(r(t)) = \nabla U(r(t)) \cdot r'(t)$ of chain rule. We will see that on Friday.

PERPETUUM MOBILES. A machine which implements a force field which is not a gradient field is called a **perpetuum mobile**. Mathematically, it implements a force fiels for which for some closed loops the energy gain is nonnegative. (By possibly changing the direction, the energy change can be made positive). The first law of thermodynamics forbids the existence of such a machine.
It is informative to stare at some of the ideas people have come up with and to see why they dont work. The drawings of Escher appear also to produce situations, where a force field can be used to get energy. He uses genious graphical tricks however.