First Midterm: Topics: (see study guide on the web)

- **vectors**. Coordinates, Operations $v+w, v-w$ and λv, dot product $(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1 w_1 + v_2 w_2 + v_3 w_3$ is scalar. $|(u \cdot v)| = |v||w|\cos(\phi)$ allows to calculate angles. cross product $v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$ is a vector orthogonal to both v and w with length $|v||w|\sin(\phi)$. Possible use: if P, Q, R are points on a plane, then $(P-Q) \times (R-Q)$ is orthogonal to the plane. Polar coordinates $(r, \phi) = (\sqrt{x^2+y^2}, \arctan(y/x))$, $(x, y) = (r\cos(\phi), r\sin(\phi))$.

- **geometry** Orthogonality $v \cdot w = 0$, parallel $v = \lambda w$, angle $\cos(\alpha) = v \cdot w/(|v||w|)$, $n = (a, b, c)$ is orthogonal to the plane $ax + by + cz = d$. Distances Point-Point $d(P, Q) = |P - Q|$,Point-Plane: $d(P, n \cdot x = d) = (P-Q) \cdot n/|n|$, Point-Line: $d(P, Q+tv) = |(P-Q) \times v|/|v|$. Line-Line: $d(P+tu, Q+tv) = |(P-Q) \cdot (v \times w)/|v \times w|$.

- **curves** $r(t) = X(t) = (x(t), y(t), z(t))$, velocity $r'(t) = X'(t) = (x'(t), y'(t), z'(t))$, speed $|X'(t)|$, lines $r(t) = P + tv$, Polar curves $(r(t), \phi(t))$ are $(x(t), y(t)) = (r(t)\cos(\phi(t)), r(t)\sin(\phi(t)))$ in Cartesian coordinates. Example of polar curves: roses $r(t) = \cos(nt), \phi(t) = t$. Length of a curve $\int_a^b |r'(t)|\, dt$. Find intersections of a curve with a plane. Find angle between curve and plane (determine angle between normal vector of plane and velocity vector of curve).

- **surfaces** $\nabla F(x, y, z)$ is normal vector to surface $F(x, y, z) = c$, curves as graphs $z = f(x, y)$, level surfaces $F(x, y, z) = c$. Examples of surfaces: planes $ax + by + cz = d$, spheres $d((x, y, z), P) = r$, $x^2 + y^2 - y^2 = r^2$ hyperboloid, $a^2x^2 + b^2y^2 + c^2z^2 = d^2$ ellipsoid, $x^2 + y^2 - z = d$ paraboloid. Find angle between intersecting surfaces (=angle between normal vectors).

- **integration, differentiation** If $r(t) = (x(t), y(t), z(t)$, then $r'(t) = (x'(t), y'(t), z'(t))$. If $F(x, y, z)$ is a function of three variables, then $\nabla F(x, y, z) = (F_x, F_y, F_z)$ is the gradient of F. $\int r(t)dt = (\int x(t)dt, \int y(t)dt, \int z(t)dt)$. Line integrals $\int F(r(t)) \cdot r'(t)\, dt$ is an energy if F is a force.

Suggested Problems:

- pgs 263-264 number 1,5,7.

REPETITION. If $F(x, y, z)$ is a vector field and $\gamma : t \mapsto r(t)$ is a curve, then $\int_a^b F(r(t)) \cdot r'(t)\, dt$ is called a **line integral** along the curve.

AN EXAMPLE. Let $X(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on the parameter interval $[0, \pi]$. Let F be the vector field $F(x, y) = (-xy, 0)$. Calculate the line integral $\int_\gamma F \cdot dX$.

SOLUTION. First, we write the curve in Cartesian coordinates: $X(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $X'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

$$\begin{aligned}
\int_0^\pi F(X(t)) \cdot X'(t)\, dt &= \int_0^\pi (\cos^4(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\, dt \\
&= -2\int_0^\pi \sin^2(t)\cos^4(t)\, dt \\
&= -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8
\end{aligned}$$

FUNDAMENTAL THEOREM OF LINE INTEGRALS. If $F = \nabla U$, then

$$\int_a^b F(r(t)) \cdot r'(t)\, dt = U(r(b)) - U(r(a))$$

the line integral gives the potential difference between the points $r(b)$ and $r(a)$.

REMEMBER THE 1D CHAIN RULE. If f and g are functions of one variables, then $d/dt f(g(t)) = f'(g(t))g'(t)$.

HIGHER DIMENSIONAL CHAIN RULE. If $r(t)$ is a curve, and $U(x, y, z)$ is a scalar field, then $\frac{d}{dt}U(r(t)) = \nabla U(r(t)) \cdot r'(t)$.

then $U(r(t))$ is the temperature, the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt}U(r(t))$.

SPECIAL SITUATIONS.
$r(t)$ parallel to level curve means $d/dtU(r(t)) = 0$ and $r'(t)$ orthogonal to $\nabla U(r(t))$
$r(t)$ orthogonal to level curve means $|d/dtU(r(t))| = |\nabla U||r'(t)|$ and $r'(t)$ parallel to $\nabla U(r(t))$.

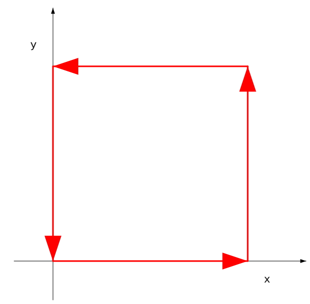
PROOF OF THE FUNDAMENTAL THEOREM. Use the chain rule in the second equation and the fundamental theorem of calculus in the third:

$$\begin{aligned} \int_a^b F(r(t)) \cdot r'(t)\, dt &= \int_a^b \nabla U(r(t)) \cdot r'(t)\, dt \\ &= \int_a^b \frac{d}{dt} U(r(t))\, dt = U(r(b)) - U(r(a)) \, . \end{aligned}$$

CALCULATING WITH CURVES.

If γ_1, γ_2 are curves, then $\gamma_1 + \gamma_2$ is the curve obtained by traveling first along γ_1, then along γ_2, write $-\gamma$ for the curve γ traveled in the opposite way.

EXAMPLES. If $\gamma_1(t) = (t, 0)$ for $t \in [0,1]$, $\gamma_2(t) = (1, (t-1))$ for $t \in [1,2]$, $\gamma_3(t) = (1-(t-2), 1)$ for $t \in [2,3]$, $\gamma_4(t) = (0, 1-(t-3))$ for $t \in [3,4]$, then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ for $t \in [0,4]$ is the path which goes around a the unit square. The path $-\gamma$ travels around in the clockwise direction.



CALCULATING WITH LINEINTEGRALS.

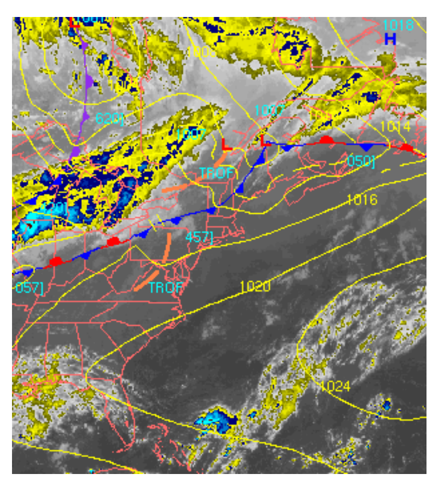
- $\int_\gamma F \cdot dX + \int_\gamma G \cdot dX = \int_\gamma (F+G) \cdot dX$.
- $\int_{\gamma_1+\gamma_2} F \cdot dX = \int_{\gamma_1} F \cdot dX + \int_{\gamma_2} F \cdot dX$
- $\int_\gamma cF \cdot dX = c \int_\gamma F \cdot dX$.
- $\int_{-\gamma} F \cdot dX = -\int_\gamma F \cdot dX$.

ABOUT PARTIAL DERIVATIVES:
$F_{xy} = F_{yx}$ holds for all smooth functions. Proof: just write down the derivatives as limits of differential quotients. Both sides are approximated by $(F(x+dx, y+dy) - F(x+dx, y) - F(x, y+dy) + F(x,y))/(dxdy)$ for small dx, dy.

WHEN IS A VECTOR FIELD A GRADIENT FIELD (2D)?

$F(x,y) = \nabla U(x,y)$ implies $F_y(x,y) = F_x(x,y)$. If this does not hold at some point, F is no gradient field.

MULTIVARIABLE CALCULUS IN WEATHER MAPS.
(Map from http://www.hpc.ncep.noaa.gov/sfc/satsfc.gif, updated every 3 hours.) Isoterms: Curves of constant temperature or pressure $p(x,y) = c$. These are level curves. Wind maps are vector fields. $F(x,y)$ is the wind velocity at the point (x,y). The wind velocity F is not quite normal to the **isobares**, the lines of equal pressure p. The scalar pressure field p and the velocity field F depend on time. The equations which describe the weather dynamics are called **Navier Stokes equations**



$$d/dtF + F \cdot \nabla F = \nu \Delta F - \nabla p + f, \mathrm{div} F = 0$$

(we will see what is Δ, div later.) It is a **partial differential equation** like $u_x - u_y = 0$ you have encountered in a homework. Finding solutions is not trivial. You can even win 1 Million dollars by proving that the equations have smooth solutions in space!