

### Suggested Problems:

- pgs 115-119 number 15,19
- pgs 124-127 number 1,3,9,15,17,19

**LINEAR APPROXIMATION. 1D:** A function of one variables  $f(x)$  is approximated at a point  $x_0$  by

$$g(x) = f(x_0) + f'(x_0)(x - x_0).$$

**2D:** A function  $f(x, y)$  is approximated at  $(x_0, y_0)$  by

$$g(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0).$$

because the level curves of  $g$  is tangent to the level curve of  $f$  at  $(x_0, y_0)$ .

**3D:** A function  $f(x, y, z)$  is approximated at  $(x_0, y_0, z_0)$  by

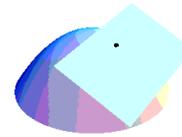
$$g(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0).$$

The level surfaces of  $g$  are planes which are tangent to the level surface of  $f$  at  $(x_0, y_0, z_0)$ .



**DEFINITION.** The function

$$g(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$



is called the **linear approximation** of  $f$  at  $\vec{x}$ .

**PHYSICAL LAWS.** Many physical laws actually are just linear approximations. One could say that a large fraction of physics consists of understand nature with linear laws.

**Linear stability analysis.** In physics, complicated situations can occur. Usually, many unknown parameters are present and the only way to model the situation is to assume that things depend linearly on these parameters. The analysis of the linear situation allows then to predict for example the stability of the system with respect to perturbations.

**Error analysis.** Error analysis is based on linear approximation. Assume, you make a measurement of a function  $F(a, b, c)$ , where  $a, b, c$  are parameters. Assume, you know the numbers  $a, b, c$  up to accuracy  $\epsilon$ . How precise do you know the values  $F(a, b, c)$ ? Because  $F(a_0 + \epsilon_a, b_0 + \epsilon_b, c_0 + \epsilon_c)$  is about  $F(a_0, b_0, c_0) + \nabla F(a_0, b_0, c_0) \cdot (\epsilon_a, \epsilon_b, \epsilon_c)$ , the answer is that we know  $F$  up to accuracy  $|\nabla F(a_0, b_0, c_0)|\epsilon$ .

**Power laws.** Some laws in physics are given by functions of the form  $f(x, y) = x^\alpha y^\beta$ . Such dependence on  $x$  or  $y$  is called **power law behavior**. If we consider  $F = \log(f)$ , and introduce  $a = \log(x), b = \log(y)$ , then this becomes  $F(a, b) = \log(f(x, y)) = \alpha a + \beta b$ . Power laws become linear laws in a logarithmic scale. They usually are linear approximations to more complicated nonlinear relations.

### EXAMPLES OF LINEARIZED LAWS.

**Ohms law.** If we apply a voltage difference  $U$  at the ends of a resistor  $R$ , then a current  $I$  flows. The relation  $U = RI$  is called **Ohms law**. In reality, the relation between current and voltage is much more complicated. If the resistor gets hot, then its characteristics begin to change. Nonlinear resistors are used in synthesizers or oscillators for example. While the Ohm law works **extremely well**, the nonlinear behavior can have important consequences for example to stabilize systems or to protect equipment against over-voltages.

**Thermal expansion.** If  $l(T)$  is the length of an object with temperature  $T$ , then  $l(T) = l(T_0) + c(T - T_0)$ , where the expansion coefficient  $c$  depends on the material. (Trick question: What happens if you heat a ring, does the inner ring become smaller or bigger). The volume of a hot air balloon and therefore, its lift capacity grows like  $c(T - T_0)^3$ . The law of expansion is only an approximation.

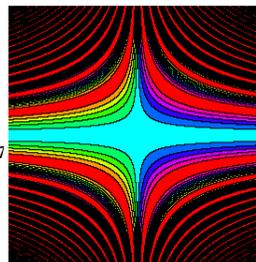
**Sea water density** For oceanographers, it is important to know the water density  $\rho(T, S)$  in dependence on the **temperature**  $T$  (Kelvin) and the **salinity**  $S$  (psu). If we would include the pressure  $P$  (Bar), then we had a function  $\rho(T, S, P)$  of three variables. Near a specific point  $(T, S, P)$  the density can be approximated by a linear function.

the nonlinear (partial) differential equations like  $u_{xx} + u_{yy} + u_{zz} = F(x, y, z)$  are often approximated by linear differential equations.

**Cartography.** It is well known that we live on a sphere. On a sphere however, a triangle has more than 180 degrees and every line (geodesics) crosses every other line at least twice. Despite this, a city map can perfectly assume that we have a Cartesian coordinate system. When drawing a plan of a house, an architect can assume that the house stands on a plane (the level curve of the linearisation  $G(x, y)$  of  $F(x, y, z)$  defining the surface of the earth.

**Relativity.** From classical mechanics we know Newton's law telling that  $r''(t)$ , the acceleration is proportional to the force  $F$  which acts on the mass point:  $r''(t) = F/m$  so that  $r'(t) = tF/m$ . This law can not apply for all times, because we can not reach the speed of light with a massive body. In special relativity, the Newton axiom is  $d/dt(r'(t)m(t)) = F$ , where the mass  $m(t)$  depends on the velocity: This gives  $v(t) = tF/m_0\sqrt{1 + (Ft/m_0c)^2}$ . Linearisation at  $t = 0$  gives  $v(t) = tF/m_0$ , the classical law.

EXAMPLE (2D). Find the linear approximation of the function  $F(x, y) = \sin(\pi xy^2)$  at the point  $(1, 1)$ . The gradient is  $\nabla F(x, y) = (\pi y^2 \cos(\pi xy^2), 2y\pi \cos(\pi xy^2))$ . At the point  $(1, 1)$ , we have  $\nabla F(1, 1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi)$ . The linear function approximating  $F$  is



$$G(x, y) = F(1, 1) + \nabla F(1, 1) \cdot (x-1, y-1) = 0 - \pi(x-1) - 2\pi(y-1) = -\pi x - 2\pi y + 3\pi$$

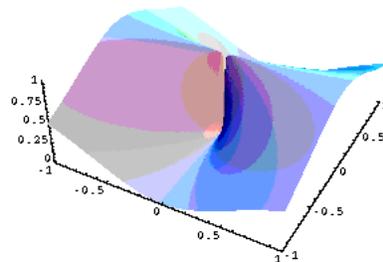
The level curves of  $G$  are lines  $x + 2y = \text{const}$ . The line passing through  $(1, 1)$  satisfies  $x + 2y = 3$ .

Application:  $-0.00943407 = F(1+0.01, 1+0.01) \sim G(1+0.01, 1+0.01) = -\pi * 0.01 - 2\pi * 0.01 + 3\pi = -0.00942478$ .

FINDING TANGENT PLANES. Geometrically, linearisation means finding the **tangent plane** to a level surface  $F(x, y, z)$ . If the linear function  $G(x, y, z)$  approximates  $F(x, y, z)$  at a point  $P$ , then the level curve of  $F$  at  $P$  and the level curve of  $G$  at  $P$  are the same. Indeed  $\nabla G(P)$  is equal to  $\nabla F$  at  $P$ . Example:  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$  is a plane tangent to the surface  $z = f(x, y)$  because  $F(x, y, z) = f(x, y) - z$  has this graph as a level curve to the level 0. The approximation at  $(x_0, y_0, f(x_0, y_0))$  gives that formula

EXAMPLE (3D). Find the tangent plane to the two-sheeted hyperboloid  $x^2 + y^2 - z^2 = -1$  at the point  $(x_0, y_0, z_0) = (2, 2, 3)$ . The hyperboloid is a level-surface to the function  $F(x, y, z) = x^2 + y^2 - z^2$ , whose gradient is  $(2x, 2y, -2z)$ . At the point  $(2, 2, 3)$ , the gradient is  $(4, 4, 6)$ . The equation of the plane is therefore  $4x + 4y - 6z = -2$ .

SMOOTHNESS. In order that linear approximation works, functions have to be smooth enough. Now, this seems to be a bit nit picking of mathematicians, but non-smooth functions play a huge role in physics phase transitions, symmetry breaking. Near such points, linear approximations usually don't work. One of the examples is  $F(x, y) = x^2/(x^2 + y^2)$ . Because  $F(x, y) = 0$  for  $x = 0$  and  $F(x, y) = 1$  for  $y = 0$ , we have  $F_x(x, y) = 0$  and  $F_y(x, y) = 0$ . Therefore, the gradient is  $(0, 0)$ . However  $F(x, x) = x^2/(2x^2) = 1/2$  for every  $x$  so that the approximation  $F(x, y) = F(0, 0) + \nabla F(0, 0) \cdot (x, y)$  is not valid.



TAYLOR EXPANSION (Preview). Later, we will see that the linear approximation  $F(x, y) = F(0, 0) + \nabla F(0, 0) \cdot (x, y)$  is just the first **Taylor approximation** of  $F$ . The next best approximation, the quadratic approximation would be, guess:  $F(x, y) = F(0, 0) + \nabla F(0, 0) \cdot (x, y) + (x, y) \cdot D^2 F(0, 0) \cdot (x, y)/2$ , where  $D^2 F(0, 0) = \begin{pmatrix} F_{xx}(0, 0) & F_{xy}(0, 0) \\ F_{yx}(0, 0) & F_{yy}(0, 0) \end{pmatrix}$  is a matrix which will play an important role in the next week. The point is that there is also for nice functions of several variables a Taylor expansion  $F(x, y) = F(0, 0) + DF(0, 0)(x, y) + D^2 F(0, 0)(x, y)^2/2 + D^3 F(0, 0)(x, y)^3/6 + \dots$ . We will however have to see what  $D^k F(x, y)(x, y)^k$  means.