10/15/2001, DIRECTIONAL DERIVATIVE **Math 21a, O. Knill**

Reminder: The second Mathematica Lab is due on October 22 and is ready to be downloaded.

Suggested Problems:

- pgs 127 number 21
- pgs 134-136 number 5,7,9

DIRECTIONAL DERIVATIVE. If F is a function of several variables and v is a vector, then $\nabla F \cdot v$ is called the **directional derivative** of F in the direction v. We write $\nabla_v F$.

$$\nabla_v F(x, y, z) = \nabla F(x, y, z) \cdot v$$

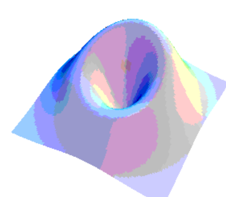
EXAMPLE. PARTIAL DERIVATIVES ARE SPECIAL DIRECTIONAL DERIVATIES. If $v = (1, 0, 0)$, then $\nabla F \cdot v = F_x$.
If $v = (0, 1, 0)$, then $\nabla F \cdot v = F_y$.
If $v = (0, 0, 1)$, then $\nabla F \cdot v = F_z$.
The directional derivative is a generalization of the partial derivatives. Like the partial derivatives, it is a **scalar**.

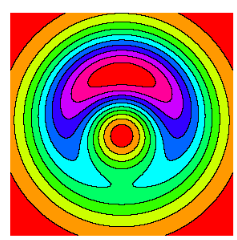
EXAMPLE. DIRECTIONAL DERIVATIVE ALONG A CURVE. If F is the temperature in a room and $r(t)$ is a curve with velocity $r'(t)$, then $\nabla F(r(t)) \cdot r'(t)$ is the temperature change, the body moving on a curve $r(t)$ experiences because we have seen that the directional derivative is in this case $d/dtF(r(t))$.

EXAMPLE: LINEAR APPROXIMATION. Assume $G(\vec{x}) = F(\vec{x_0}) + \nabla F(\vec{x_0}) \cdot (\vec{x} - \vec{x_0})$ is the linear approximation of F near a point $\vec{x_0}$. Lets see what this means if $\vec{x}$ lies on a line $\vec{x} = \vec{x_0} + t\vec{v}$. The function $f(t) = F(\vec{x} + t\vec{v})$ is then a function of one variable, t. Also $g(t) = F(\vec{x_0}) + tD_v(\vec{x_0})$ is a function of t. The function g is just the linear approximation of f and $g' = D_v F$ at $\vec{x_0}$.

The linear approximation G of F at (x_0, y_0) satisfies $D_v F(x_0, y_0) = D_v G(x_0, y_0)$ for all v.

RECALL. Gradients are orthogonal to the level surfaces. Do you remember why? The level surface of the linear approximation is tangent at the point and the tangent surface $\nabla F \cdot (x - x_0)$ has every vector $x - x_0$ in the surface orthogonal to ∇F.





STEEPEST DECENT. The directional derivative satisfies

$$|D_v F| \leq |\nabla F||v|$$

because $\nabla F \cdot v = |\nabla F||v|| \cos(\phi)| \leq |\nabla F||v|$. The direction $v = \nabla F$ is the direction, where F increases most, the direction $-\nabla F$ is the direction of steepest decent.

IN WHICH DIRECTION DOES F INCREASE? If $v = \nabla F$, then the directional derivative is $\nabla F \cdot \nabla F = |\nabla F|^2$. This means that F **increases**, if we move into the direction of the gradient!

EXAMPLE. You are on a trip in a zeppelin at $(1,2)$ and want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x,y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?

Parameterize the direction by $v = (\cos(\phi), \sin(\phi))$. The pressure gradient is $\nabla p(x,y) = (2x, 4y)$. The directional derivative in the ϕ-direction is $\nabla p(x,y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

ZERO DIRECTIONAL DERIVATIVE. The rate of change in every direction is zero if and only if $\nabla F(x,y) = 0$ because if ∇F is not 0, we can choose $v = \nabla F$ and get $D_{\nabla F} F = |\nabla F|^2$.

We will see later that points with $\nabla F = 0$ are candidates for local maxima or minima of F. Points (x,y) where $\nabla F(x,y) = (0,0)$ are called **stationary points** or **critical points**. Knowing the critical points is important to understand the function F.

LINE INTEGRALS AND RATE OF CHANGE. A reformulation for lineintegrals of conservative fields:

The line integral $\int_a^b \nabla F(r(t)) \cdot r'(t)\, dt$ integrates up the rate of change F along the curve.

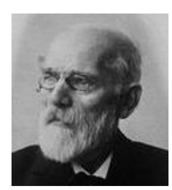
PROPERTIES OF THE DIRECTIONAL DERIVATIVE. The directional derivative D_v has all the properties of all derivatives:

$$D_v(F+G) = D_v(F) + D_v(G)$$
$$D_v(FG) = D_v(F)G + FD_v(G)$$

THE MATTERHORN is a popular climbing mountain in the Swiss alps. Its height is 4478 meters (14,869 feet). It is quite easy to climb with a guide. There are ropes and ladders at difficult places. Even so, about 3 people die each year from climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $F(x,y)$ of the Matterhorn is approximated by $F(x,y) = 4000 - x^2 - y^2$. At height $F(-10,10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the north-east direction $v = (1,-1)$. Calculate the rate of change in that direction. We have $\nabla F(x,y) = (-2x, -2y)$, so that $(20,-20) \cdot (1,-1) = 40$. This is a place, with a ladder, where you climb 40 meters up when advancing 1m forward.

THE VAN DER WAALS (1837-1923) equation for real gases is

$$(p + a/V^2)(V - b) = RT(p,V) \ ,$$



where $R = 8.314 J/Kmol$ is a constant called the Avogadro number. This law relates the pressure p, the volume V and the temperature T of a gas. The constant a is related to the molecular interactions, the constant b to the finite restvolume of the molecules.

The **ideal gas** law $pV = nRT$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(p,V) = const$ tell much about the properties of the gas. The so called **reduced van der Walls law** $T(p,V) = (p + 3/V^2)(3V-1)/8$ is obtained by scaling p, T, V depending on the gas. Calculate the directional derivative of $T(p,V)$ at the point $(p,V) = (1,1)$ into the direction $v = (1,2)$. We have $T_p(p,V) = (3V-1)/8$ and $T_V(p,V) = 3p/8 - (9/16)1/V^2 - 3/(4V^3)$. Therefore, $\nabla T(1,1) = (1/4, 0)$ and $D_v T(1,1) = 1/4$.