

Suggested Problem: pg 92 numbers 16,18 pgs 395 numbers 3,5

• pg 127 number 20,22

• pg 146 number 6

Homework: • pgs 134-136 number 2,8,10,12,14,16

• pg 395 number 2,4,8

• pg 140 number 2

• pg 93 number 28

- The function $u(x, y) = x^2 - y^2$ obeys the partial differential equation $u_x x + u_y y = 0$. Write down two more non-constant solutions to this same equation, neither a constant multiple of $x^2 - y^2$.

WHAT IS THE MATRIX? "Nobody can tell you what the matrix is, you have to find it out by yourself." (Morpheus, in the "The Matrix")



MATRIX. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ or $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ are **matrices**.

MULTIPLICATION WITH VECTOR. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix and $v = (v_1, v_2)$ is a vector, define a new vector $Av = (a_{11}v_1 + a_{12}v_2, a_{21}v_1 + a_{22}v_2)$. The vector Av is called the image of v under A .

$$Av = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}$$

ADDITION AND MULTIPLICATION OF MATRICES. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ are 2×2 matrices, then their sum is $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$ and their product is $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

EXAMPLE.

$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, \quad B = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}.$$

$$AB = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \text{ (rotations)}$$

WHERE DO MATRICES APPEAR IN MULTIVARIABLE CALCULUS?

- The Hessian matrix $A = F''(x, y)$ (2×2 matrix) or $A = F''(x, y, z)$ (3×3 matrix).
- Matrices appear in change of variables when integrating functions of several dimensions.
- Matrices appear when studying parameterized surfaces $(u, v) \mapsto (f(u, v), g(u, v), h(u, v))$.

DETERMINANT. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the **determinant** $\det(A)$ of A is defined as $a_{11}a_{22} - a_{21}a_{12}$. If $v = (a_{11}, a_{21}, 0)$ and $w = (a_{12}, a_{22}, 0)$ are the **column vectors** of A , then $v \times w = (0, 0, \det(A))$ and $\det(A)$ is the area of the parallelogram spanned by v and w .

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [a_{11}a_{22} - a_{12}a_{21}]$$

For a 3×3 matrix A with column vectors u, v, w , define $\det(A) = u \cdot v \times w$. It is the volume of the parallelepiped spanned by u, v, w . The determinant can be defined for $n \times n$ matrices.

$$+ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$- \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

TRACE. The sum of the diagonal elements of a matrix is called the **trace** $\text{Tr}(A)$.

$$\text{Trace} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} + a_{22} + a_{33}$$

EIGENVALUES. The roots of the function $\lambda \mapsto \det(\lambda I - A)$ are called **eigenvalues** of A .

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = \lambda^2 + (a + d)\lambda + ad - bc = \lambda^2 + \text{tr}(A)\lambda + \det(A)$. The roots are $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$.

IN MATHEMATICA.

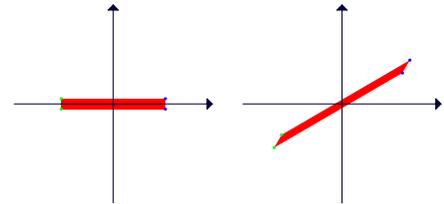
$A = \{\{1, 3\}, \{4, 6\}\}$
 $B = \{\{2, 4\}, \{2, -6\}\}$
 $v = \{4, 5\}$

$A.v, A.B, A + B$
 Eigenvalues[A]
 Det[A]

$\text{Tr}[A]$.
 Eigensystem[A]
 $A[[1, 2]]$ (entry a_{12} of the matrix)

RELATIVITY. The **Galileo transformation** $x' = x - vt$ can be written as $\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$. Relativity replaces this with the **Lorentz boost**: $\begin{bmatrix} t' \\ x' \end{bmatrix} = (1/\gamma) \begin{bmatrix} 1 & -v/c^2 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$, where c is the speed of light and $\gamma = \sqrt{1 - v^2/c^2}$. With $\epsilon = 1/c^2$, the Galileo transformation is just a linearisation of the Lorentz transformation at $\epsilon = 0$. The boost, is actually **beautiful**: if $\tanh(\phi) = v/c$, trigonometric identities $\sinh(x) = \tanh(x)/\sqrt{1 - \tanh(x)^2}$, $\cosh x = 1/\sqrt{1 - \tanh(x)^2}$ give $\sinh(\phi) = (v/c\gamma)$, $\cosh(\phi) = 1/\gamma$. The boost can be written as $\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} 1/\gamma & -v/(c\gamma) \\ -v/(c\gamma) & 1/\gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$. It is a **hyperbolic rotation**.

LORENTZ CONTRACTION. If a bar $[a, b]$ moves with speed v , its position is $[a + tv, b + tv]$, classically. Relativistically, its endpoints are at $[(a + tv)/\gamma, (b + tv)/\gamma]$: the bar is by a factor $1/\gamma$ larger in the moving system - or - it is by γ smaller in the fixed system. This is the **Lorentz contraction**. If $v = 2/3c$, the contraction is already 75 percent. The figure shows the bar in the moving coordinate system and then in the fixed coordinate system, where the two end points of the bar have a different time. (If a light signal were sent out simultaneously at both ends, then the signal to the left would reach 0 earlier than the signal starting to the right.)



ABOUT DETERMINANTS IN PHYSICS. Determinants are important in physics. For example in "quantisation procedure", where Feynman path integrals are "summations" over a suitable set of "paths". What does it have to do with determinants? Each summand of the determinant can be interpreted as a contribution of a path in a finite graph with n nodes.

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Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.



The Ray-Singer determinant is a number attached to a geometry. Physicists trying to glue quantum mechanics with general relativity hope to make sense of expressions like $\int_g e^{i \det(A(g))}$, where A is a function of the geometry g and where the integral "sums" over all possible geometries.