Suggested Problem: pags 145-146, number 3 (no technology).

SECOND DERIVATIVE.

If $F(x,y)$ is a function of two variables, then the matrix $F''(x,y) = \begin{pmatrix} F_{xx}(x,y) & F_{xy}(x,y) \\ F_{yx}(x,y) & F_{yy}(x,y) \end{pmatrix}$ is called the **second derivative** or the **Hessian** of F at (x,y).

For a function F of three variables, the Hessian at (x,y,z) is the 3×3 matrix $F''(x,y,z) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$ (each entry i.e. $F_{xy}(x,y,z)$) depends on the point (x,y,z).

RECALL. For smooth functions, $F_{xy} = F_{yx}$, $F_{xz} = F_{zx}$ etc. The matrix F'' is therefore **symmetric**. This means that reflecting it at the diagonal gives the same matrix.

PROOF OF $F_{xy} = F_{yx}$. Compare
$F_{xy}dxdy \sim [F(x+dx,y+dy) - F(x,y+dy)] - [F(x+dx,y) - F(x,y)]$
$F_{yx}dydx \sim [F(x+dx,y+dy) - F(x+dx,y)] - [F(x,y+dy) - F(x,y)]$.

QUADRATIC APPROXIMATION. If F is a function of several variables $\vec{x}$ and $\vec{x}_0$ is a point, then

$$Q(\vec{x}) = F(\vec{x}_0) + \nabla F(\vec{x}_0)(\vec{x}-\vec{x}_0) + [F''(\vec{x}_0)(\vec{x}-\vec{x}_0)] \cdot (\vec{x}-\vec{x}_0)/2$$

is called the **quadratic approximation** of $\vec{x}$. It generalizes the Taylor approximation $g(x) = f(y) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$. of a function of one variables.

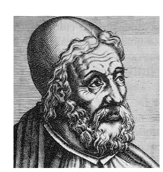
CUBIC APPROXIMATION. The approximation can be pushed further: if F is a function of several variables $\vec{x} = (x^1, ..., x^j)$ and $\vec{y}$ is a point, then (we write $(\vec{x}-\vec{y})^j$ for the j'th component of the vector $(\vec{x}-\vec{y})$):

$$C(\vec{x}) = F(\vec{y}) + \sum_i F_{x^i}(\vec{y})(\vec{x}-\vec{y})^j + \sum_{i,j} \frac{F_{x^i x^j}(\vec{y})(\vec{x}-\vec{y})^i(\vec{x}-\vec{y})^j}{2} + \sum_{i,j,k} \frac{F_{x^i x^j x^k}(\vec{y})(\vec{x}-\vec{y})^i(\vec{x}-\vec{y})^j(\vec{x}-\vec{y})^k}{6}$$

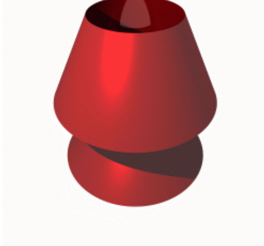
is called the **cubic approximation** of $\vec{x}$. We will never compute such an approximation by hand in Math21a. The multi-dimensional cubic approximation generalizes the cubic Taylor approximation $g(x) = f(y) + f'(y)(x - y) + f''(y)\frac{(x-y)^2}{2} + f'''(y)\frac{(x-y)^3}{6}$ in one variable. $F_{x^i x^j x^k}$ is an example of a "tensor" (an object which generalizes vectors and matrices).

WHY DO WE DO APPROXIMATIONS AT ALL?
• In general, a computer can not give the precise value of a function and approximates it with a polynomial. A calculator might for example calculate $x - x^3/3! + x^5/5! - x^7/7! + x^9/9!$ instead of $\sin(x)$.
• Low order approximations like linear or quadratic approximations lead to **simple relations** or **physical laws**, often accurate enough for applications.
• The quadratic approximation is important when trying to classify **critical points** (see next week).
• Approximations in the complex plane are actually **Fourier approximations** (see Math21b) $f(z) = a+bz+cz^2$ has a real part $a+b\cos(t)+c\cos(2t)$. Quadratic approximation would correspond to an approximation using 2 frequencies. Fourier approximations play a role in compression schemes like JPG or MP3. The eye or the ear does not distinguish between a good approximation and the real thing.

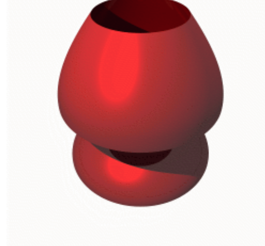
• Approximations are sometimes so accurate, that scientists are mislead and think that "it is it": examples are the Ptolemaic **epicycle approximation** of the planetary motion (which are essentially quadratic, cubic etc approximations of the ellipse when written using complex numbers). Other examples are classical mechanics ($c = \infty$ approximation) of special relativity, or $\hbar = 0$ approximation of quantum mechanics.



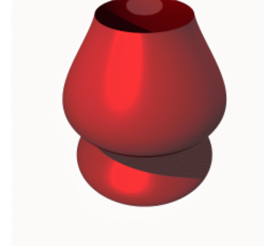
APPROXIMATIONS IN COMPUTER GRAPHICS. When a graphic artist or designer draws a curve or surface, he or she patches together simple surfaces. In the simplest case, these are flat triangularizations. The next best approximation would be to put together pieces of quadrics. More refined are cubic patches. More advanced constructions exist like NURBS (nonuniform rational B splines). The pictures illustrates linear quadratic and cubic approximations for **lathes**, rotational symmetric surfaces.



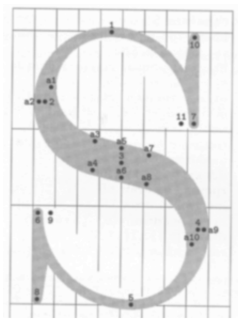
using a linear spline



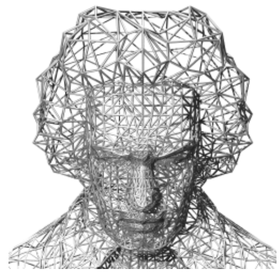
using a quadratic spline



using a cubic spline



Letters consist of quadratic or cubic splines. (see Donald E. Knuth book "Digital Typography")



3D graphics handles surfaces by approximations from a set of points and connections.

A simple approximation of the Beethoven surface is obtained by piecewise linear triangularisations.

BACK TO THE MATTERHORN. The height of the Matterhorn is given by $F(x,y) = 4000 - \sin(x^2 - x + 2y^2)$. Find the quadratic approximation of F at $(0,0)$.
$\nabla F(x,y) = (-2x+1, -4y)\cos(x^2 - x + 2y^2)$ so that $\nabla F(0,0) = (1,0)$. The linear approximation of F at $(0,0)$ is $G(x,y) = F(0,0) + \nabla F(0,0) \cdot (x,y) = 4000 + x$. The graph of G is a plane tangent to the graph of F.

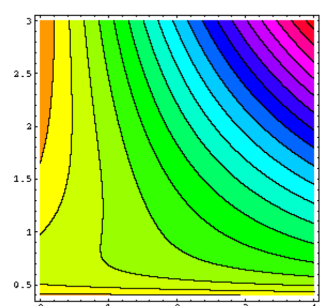
$$F''(x,y) = -\cos(x^2 - x + 2y^2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

so that $F''(0,0) = -\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $[F''(0,0)(x-x_0, y-y_0)] = (x - x_0, 2(y - y_0)) = -(x, 2y)$ and $[F''(0,0)(x-x_0,y-y_0)] \cdot (x-x_0, y-y_0)/2 = -x^2 - 2y^2$. The quadratic approximation at $(0,0)$ is $Q(x,y) = 4000 + x - x^2 - 2y^2$. The graph of F is an inverted distorted paraboloid.

BACK TO THE **REDUCED VAN DER WAALS LAW** (from Monday).

$$T(p,V) = (p + 3/V^2)(3V-1)/8$$

Find the quadratic approximation of T at $(p,V) = (1,1)$. We had $T_p(p,V) = (3V-1)/8$ and $T_V(p,V) = 3p/8 - (9/8)1/V^2 + 3/(4V^3)$ and $T'(1,1) = \nabla T(1,1) = (1/4, 0)$. Now, $T_{pp}(p,V) = 0$, $T_{pV} = 3/8$, $T_{Vp} = 3/8$ and $T_{VV} = (9/4)1/V^3 - (9/4)1/V^4$ so that $T''(1,1) = \begin{pmatrix} 0 & 3/8 \\ 3/8 & 0 \end{pmatrix}$. The quadratic approximation of $T(p,V)$ at $(p,V) = (1,1)$ is $Q(p,V) = T(1,1) + T'(1,1) \cdot (p-1, T-1) + [T''(1,1)(p-1,T-1)] \cdot (p-1,T-1)/2$.



$$Q(p,V) = 1 + (p-1)/4 + 3/8(p-1)(V-1)$$

MATHEMATICA KNOWS TO DO QUADRATIC APPROXIMATIONS ALSO: $T[p\_, V\_] := (p + 3/V^2)(3V - 1)/8$; **Series**$[T[p,V], \{p,1,2\}, \{V,1,2\}]$