10/22/2001, CRITICAL POINTS **Math 21a, O. Knill**

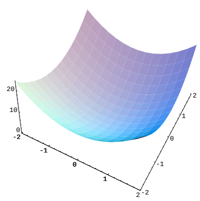
Suggested Problem:

- pags 156-159, number 3,11,13,19
- Consider the function of two variables (p, θ) given by $p^2/2 + \cos(\theta)$, where p can be any real number and $0 \leq \theta \leq 2\pi$. This is the energy of the pendulum. Find the minima of this function.

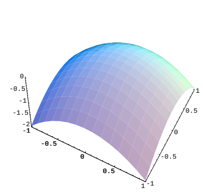
CRITICAL POINTS. A point (x_0, y_0) is called a critical point of $F(x, y)$ if $\nabla F(x_0, y_0) = 0$. A point (x_0, y_0, z_0) is called a critical point of $F(x, y, z)$ if $\nabla F(x_0, y_0, z_0) = 0$.

EXAMPLE1. $F(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla F(x, y) = (2x\cos(x^2 + y), \cos(x^2 + y) + 1)$. Where are the critical points? We must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $(0, \pi), (0, 3\pi)$.
EXAMPLE2. ("volcano") $F(x, y) = (x^2 + y^2)e^{-x^2-y^2}$. The gradient $\nabla F = (2x - 2(x^2 + y^2), 2y - 2(x^2 + y^2))e^{-x^2-y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$.

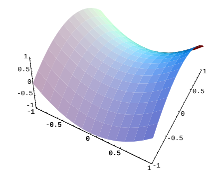
TYPICAL EXAMPLES.



$F(x, y) = x^2 + y^2$.



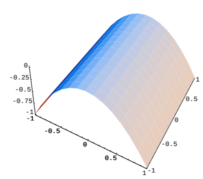
$F(x, y) = -x^2 - y^2$.



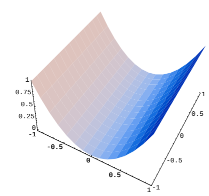
$F(x, y) = x^2 - y^2$.

EXAMPLES WITH $\det(H) = 0$.

4) $F(x, y) = x^2$.

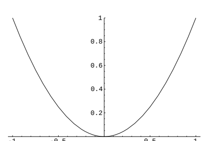


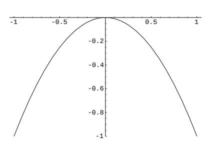
5) $F(x, y) = -x^2$.

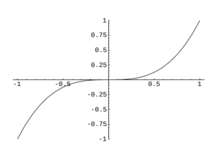


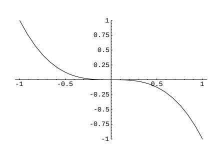
CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSIONS.

$f'(x) = 0, f''(x) > 0$, local minimum, $f''(x) < 0$ local maximum, $f'' = 0$.







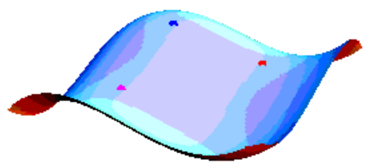


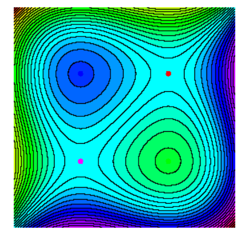
CLASSIFICATION OF CRITICAL POINTS FOR FUNCTIONS OF TWO VARIABLES.
Let $F(x, y)$ be a function of two variables with critical point (x_0, y_0). Let H be the Hessian at (x_0, y_0).

$\det(H) > 0$ $\mathrm{tr}(H) > 0$ local minimum (valley)	$\det(H) = 0$, $\mathrm{tr}(H) < 0$ mountain line
$\det(H) > 0$ $\mathrm{tr}(H) < 0$ local maximum (mountain top).	$\det(H) = 0$, $\mathrm{tr}(H) > 0$ long valley.
$\det(H) < 0$ saddle point (mountain pass).	$\det(H) = 0$, $\mathrm{tr}(H) = 0$ need higher derivatives.

EXAMPLE. (A "napkin").
The function $F(x,y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla F(x,y) = (x^2-1, -y^2+1)$. It vanishes at the 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix is $H = F''(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

$F''(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$F''(-1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$F''(1,-1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$F''(-1,-1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$\det(H) = -4$ Saddle point	$\det(H) = 4, \mathrm{tr}(H) = -4$ Local maximum	$\det(H) = 4, \mathrm{tr}(H) = 4$ Local minimum	$\det(H) = -4$ Saddle point



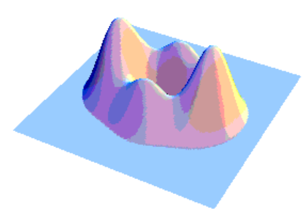


PROOF. The behavior of F near (x_0, y_0) is the same as the behavior of the quadratic approximation Q. By translation of the origin, we have $(x_0, y_0) = (0,0)$ and by translation of F we can assume $F(0,0) = 0$. We can therefore assume $F(x,y) = ax^2 + bxy + cy^2$. After a rotation of the coordinate system, we get $F(x,y) = \lambda_1 x^2 + \lambda_2 y^2$, where λ_i are the eigenvalues of H, (the roots of $\det(\lambda - H) = \lambda^2 - \mathrm{tr}(H)\lambda + \det(H)$.) The product $\lambda_1\lambda_2$ is $\det(H)$ and $\lambda_1 + \lambda_2 = \mathrm{tr}(H)$.

MAXIMA-MINIMA. To determine the maximum or minimum of a function $F(x,y)$ on a domain, we determine all critical points **inside the domain**, and compare their values with maxima or minima **on the boundary**.

WHY DO WE CARE ABOUT CRITICAL POINTS?
- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature determine a lot about the function.
- Critical points are physically relevant (i.e. configurations with lowest energy).

A CURIOUS OBSERVATION: (The island theorem) Let $F(x,y)$ be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ "charge" if it is a maximum or minium, and with -1 "charge" if it is a saddle point. Sum up all the charges and you will get 1, independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



CRITICAL POINTS IN PHYSICS. (informative) As a motivation for critical point analysis, one should mention that most physical laws are based on a simple principle: the equations are critical points of some function in infinite dimensions. **Examples:**
- **Newton equations**. (Classical mechanics)If a particle of mass m moves in a field V along a path $\gamma : t \mapsto r(t)$, consider the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$. Critical points γ satisfy $mr''(t)/2 - \nabla V(r(t)) = 0$ which are Newtons equations.
- **Maxwell equations**. (Electromagnetism) If $(E(t), B(t))$ are the electric and magnetic field, consider the Integral $S(E,V) = \frac{1}{8\pi}\int (E^2 - B^2)\, dV$ over space time. Critical points satisfy the Maxwell equations in vacuum.
- **Einstein equations** (General relativity) If g is a dot product which depends on space and time, and R is the "curvature" of the corresponding curved space time, then $S(g) = \int R dV(g)$ is a function of g for which critical points g satisfy the Einstein equations in general relativity.