Homework:

- pgs 156-159 numbers 2,8,10,16,20,26ac

- pgs 245-248 numbers 2,4,6,8

- pgs 168-169 numbers 4,8,12,14

- Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.

- Find the maximum and minimum values of $F(x, y, z) = x + y - 2z$ on the sphere $x^2 + y^2 + z^2 = 1$.

- Find the local extreme points of $F(x, y, z) = xyz$ on the surface $x + y + 4z = 1$.

- Model the earth as the sphere $x^2 + y^2 + z^2 = 1$. Suppose that the temperature at a point (x, y, z) on the surface is $T(x, y, z) = x^2 - y^2 + z + 1$ in appropriate units. Find the points with the highest and lowest temperatures.

- Suppose that the profit from Scooter sales is a function N of variables (x, y, z) that is given by $N(x, y, z) = -4x^2 + 2xy - z^2$. Suppose as well that the values of (x, y, z) are not independent, but constrained by $x + y + 2z = 1$. What values of (x, y, z) should be used to maximize the profit?

- Let $c > 0$ be a constant. An example of a **wave equation** is the partial differential equation $u_{tt} - c^2 u_{xx} = 0$ for a function $u(t, x)$. If $f(s)$ and $g(s)$ are any pair of one variable functions, use the chain rule to prove that $u(t, x) = f(x - ct) + g(x + ct)$ solves the wave equation.

Suggested Problems:

- pages 157-169 numbers 21 and 23

- pages 245-248 numbers 1,3,7

- answered problems in the Lagrange multiplier handout.

CONSTRAINED EXTREMA. Given a function $F(x, y)$ of two variables and a curve $G(x, y) = c$. Find the extrema of F on the curve.

You see that at places, where the gradient of F is not parallel to the gradient of G, the function F changes when we change position on the curve $G = c$.

Therefore we must have a solution of three equations

$$\nabla F(x, y) = \lambda \nabla G(x, y), G(x, y) = c$$

to the three unknowns (x, y, λ). The constant λ is called the **Lagrange multiplier**. The equations obtained from the gradients are called **Lagrange equations**.
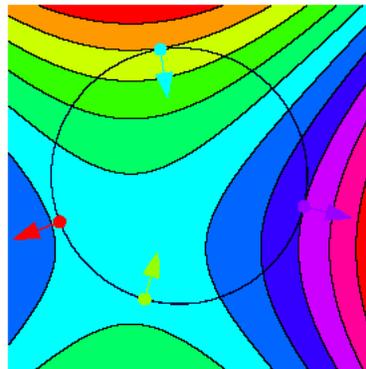
HIGHER DIMENSIONS. The above contrained extrema problem works also in more dimensions. For example, if $F(x, y, z)$ is a function of three variables and $G(x, y, z) = c$ is a surface, we solve the system of 4 equations

$$\nabla F(x, y, z) = \lambda \nabla G(x, y, z), G(x, y) = c$$

to the 4 unknowns (x, y, z, λ). In n dimensions, we have $n + 1$ equations and $n + 1$ unknowns $(x_1, \ldots, x_n, \lambda)$.

EXAMPLE. Extrema of $F(x, y, z) = z$ on the sphere $G(x, y, z) = x^2 + y^2 + z^2 = 1$ are obtained by calculating the gradients $\nabla F(x, y, z) = (0, 0, 1)$, $\nabla G(x, y, z) = (2x, 2y, 2z)$ and solving $(0, 0, 1) = \nabla F = \lambda \nabla G = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The extrema are the north pole $(0, 0, 1)$ (maximum) and the south pole $(0, 0, -1)$ (minimum).

THE PRINCIPLE OF MAXIMAL ENTROPY. Consider a dice showing i with probability p_i, where $i = 1, ..., 6$. The **entropy** of the probability distribution is defined as $S(\vec{p}) = -\sum_{i=1}^{6} p_i \log(p_i)$. Find the distribution p which maximizes entropy under the constrained $G(\vec{p}) = \sum_{i=1}^{6} p_i = 1$.

SOLUTION: $\nabla F = (-1 - \log(p_i))$, $\nabla G = (1, \ldots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, \sum_i p_i = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = n\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/n) - 1$ so that $p_i = 1/n$. The distribution, where each event has the same probability is the distribution of maximal entropy.
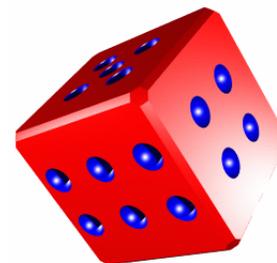
REMARK. Maximal entropy means **least information content**. A dice which is fixed (asymetric weight distribution for example) allows a cheating gambler to gain profit. Cheating through asymetric weight distributions can be avoided by making the dices transparent.


THE PRINCIPLE OF MINIMAL FREE ENERGY. Assume that the probability that a system is in a state i is p_i and that the energy of the state i is E_i. By a fundamental principle, nature tries to minimize the **free energy** $F(x, y, z) = -\sum_i (p_i \log(p_i) - E_i p_i)$ when the energies E_i are fixed. The **free energy** is the difference of the **entropy** $S(p) = -\sum_i p_i \log(p_i)$ and the **energy** $E(p) = \sum_i E_i p_i$. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called the **Gibbs distribution**.

SOLUTION: $\nabla F = (-1 - \log(p_1) - E_1, \ldots, -1 - \log(p_n) - E_n)$, $\nabla G = (1, \ldots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1 - \lambda)$. The additional equation $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The Gibbs solution is $p_i = \exp(-E_i)/\sum_i \exp(-E_i)$. For example, if $E_i = E$, then the Gibbs distribution is the uniform distribution $p_i = 1/n$.

MOST ECONOMIC ALUMINIUM CAN. You manufacture cylindrical diet coke cans of height h and radius r. You want for a fixed volume $V(r, h) = h\pi r^2 = 1$ a minimal surface area $A(r, h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $F(x, y) = 2xy + 2\pi y^2$ under the constrained $G(x, y) = xy^2 = 1$. Calculate $\nabla F(x, y) = (2y, 2x + 4\pi y), \nabla G(x, y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$
REMARK. Other factors influence the shape also. i.e. it has to withstand pressure forces up to 100 psi.

TWO CONSTRAINTS. (informal) The calculation with Lagrange multipliers can be generalized: if the goal is to optimize a function $F(x, y, z)$ under the constraints $G = c$ $H = d$, take the Lagrange equations $\nabla F = \lambda \nabla G + \mu \nabla H, G = c, H = d$, which are 5 equations for the 5 unknowns (x, y, z, λ, μ). Geometrically the gradient of F is in the plane spanned by the gradients of G and H (this is the plane orthogonal to the curve $G = c, H = d$.)
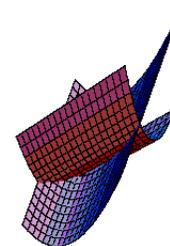
WHERE DO CONSTRAINED EXTREMAL PROBLEMS OCCUR. (informal)

- In physics, we often have conserved quantitites (like for example energy, or momentum, or angular momentum), constrainets occur naturally.

- In economical situations, one often has to optimize things under constraints (there is no free lunch).

- In mechanics constraints occur naturally, a robotarm can move for example only on a prescribed surface.

- Constrained optimization is important in statistical mechanics, where equilibria have to be obtained under constraints (i.e. energy).

- Probability distributions are solutions of constrained problems. For example, the Gaussian distribution is the distribution on the line with maximal entropy.

- Eigenvalue problems (see Math21b) can be interpreted as contrained problems. Maximize $u \cdot Lu$ under the condition $|u|^2 = 1$ gives the equation $Lu = \lambda u$ and u has to be an eigenvector. The Lagrange multiplier is an eigenvalue.