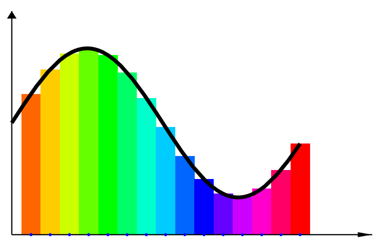
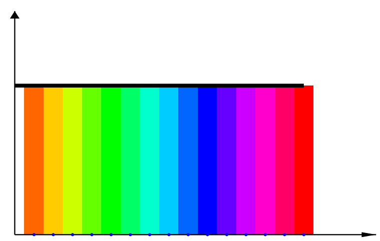
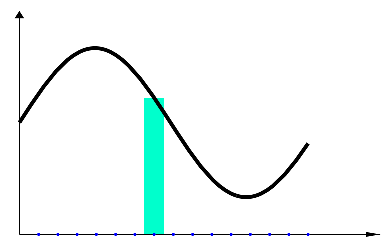
Suggested Problems:

- pages 179-182, numbers 5a,c, 7,9,11
- pages 191-193 number 1 (no technology)

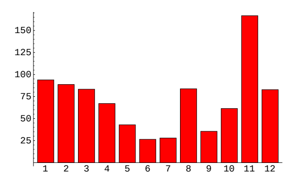
1D INTEGRATION IN 100 WORDS. If $f(x)$ is a function of one variable, then $\int_a^b f(x)\ dx$ is defined as a limit of the **Riemann sum** $f_n(x) = \frac{1}{n}\sum_{k=1}^n f(x_k)$ for $n \to \infty$ with $x_k = k/n$). The integral is the **average** of f on the interval $[a,b]$. It can be interpreted as an **signed area** under the graph of f. If $f(x) = 1$, the integral is the **length** of the interval. The function $F(x) = \int_a^x f(y)\ dy$ is called the **anti-derivative** of f. The fundamental theorem of calculus states $F'(x) = f(x)$. Unlike the derivative, anti-derivatives can not always be expressed in terms of known functions: Example: $F(x) = \int_0^x e^{-x^2}\ dx$. Often, we can find the anti-derivative: Example: $f(x) = \sin^2(x) = (\cos(2x)+1)/2, F(x) = x/2 - \sin(2x)/4$.

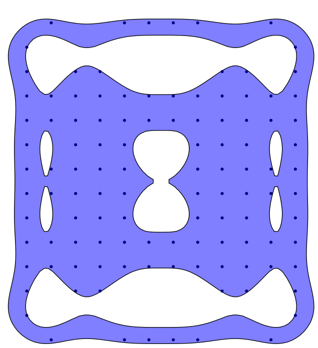
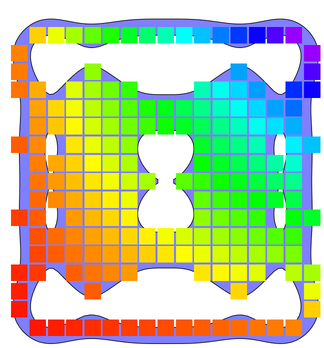
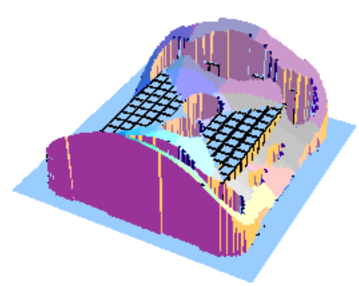
AVERAGES=MEAN. www.worldclimate.com gives the following data for the average monthly rainfall (in mm) for Cambridge, MA, USA (42.38 North 71.11 West,18m Height).

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
93.9	88.6	83.3	67.0	42.9	26.4	27.9	83.8	35.5	61.4	166.8	82.8

The average $860.3/12 = 71.7$ is a Rieman sum integral.



2D INTEGRATION. If $f(x,y)$ is a function of two variables on some region R, the integral $\int_R f(x,y)\ dxdy$ is defined as the limit $\frac{1}{n^2}\sum_{i,j,x_{i,j}\in R} f(x_i,y_j)$ with $x_{i,j} = (i/n, j/n)$ for $n \to \infty$. The integral is the **average** value of f on the region R. If $f(x,y) = 1$, then the integral is the **area** of the region R. For a few regions, the integral can be calculated as a **double integral** $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\ dy]dx$. In general, the region must be split into regions, where we can do so and add the pieces up.

EXAMPLE. Calculate $\int_R f(x,y)\ dxdy$, where $f(x,y) = 4x^2y^3$ and R is the rectangle $[0,1]\times[0,2]$.

$$\int_0^1 [\int_0^2 4x^2y^3\ dy]\ dx = \int_0^1 [x^2y^4|_0^2]\ dx = \int_0^1 x^2(16-0)\ dx = 16x^3/3|_0^1 = \frac{16}{3}\ .$$

EXAMPLE. Let R be the triangle $x \geq 0, y \geq 0, y \leq x$. Calculate $\int_R e^{-x^2}\, dxdy$.
ATTEMPT. $\int_0^1 [\int_y^1 e^{-x^2}\, dx] dy$. We can not solve the inner integral because e^{-x^2} has no anti-derivative in terms of elementary functions.
IDEA. Switch order: $\int_0^1 [\int_0^x e^{-x^2}\, dy]\, dx = \int_0^1 x e^{-x^2}\, dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316....$

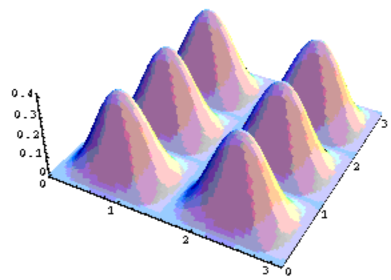
If you can't solve a double integral, try to change the order of integration!

QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function $u(x,y)$, the wave function. Unlike in classical mechanics, the position of a particle is given probabilistic way. If R is a region and u is normalized that $\int |u|^2\, dxdy = 1$, then $\int_R |u(x,y)|^2\, dxdy$ is the **probability**, that the particle is in R.

EXAMPLE. Unlike classicle particles, quantum particles in a box $[0,\pi] \times [0,\pi]$ can have a discrete set of energies only. This is the origin for the name "quantum". If $-(u_{xx} + y_{yy}) = \lambda u$ a particle of mass m has the energy $E = \lambda \hbar^2/2m$. A function $u(x,y) = \sin(kx)\sin(ny)$ represents a particle of energy $(k^2+n^2)\hbar^2/(2m)$. What is the probability that the particle with energy $13\hbar^2/(2m)$ is in the middle 9'th of the box?

SOLUTION: We first have to normalize $u^2(x,y) = \sin^2(2x)\sin^2(3y)$, so that the average over the whole square is 1:

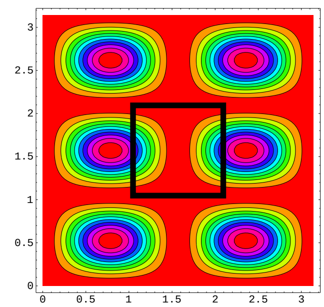
$$A = \int_0^\pi \int_0^\pi \sin^2(2x)\sin^2(3y)\, dxdy\,.$$



To calculate this integral, we first determine the inner integral $\int_0^\pi \sin^2(2x)\sin^2(3y)\, dx = \sin^2(3y)\int_0^\pi \sin^2(2x)\, dx = \frac{\pi}{2}\sin^2(3y)$ (the factor $\sin^2(3y)$ is treated as a constant). Now, $A = \int_0^\pi (\pi/2)\sin^2(3y)\, dy = \frac{\pi^2}{4}$, so that the **probability amplitude function** is $f(x,y) = \frac{4}{\pi^2}\sin^2(2x)\sin^2(3y)$.

The probability that the particle is in $R = [\pi/3, 2\pi/3] \times [\pi/3, 2\pi/3]$ is

$$\begin{aligned}\int_R f(x,y)\, dxdy &= \int_{\pi/3}^{2\pi/3}\int_{\pi/3}^{2\pi/3} \sin^2(2x)\sin^2(3y)\frac{4}{\pi^2}\, dxdy \\ &= \frac{4}{\pi^2}(4x - \sin(4x))/8|_{\pi/3}^{2\pi/3}(6x - \sin(6x))/12|_{\pi/3}^{2\pi/3} \\ &= 1/9 - 1/(4\sqrt{3}\pi)\end{aligned}$$



The probability is slightly smaller than 1/9.

MOMENT OF INERTIA. Compute the kinetic energy of a square iron plate $R = [-1,1] \times [-1,1]$ of density $\rho = 1$ (about 10cm thick) rotating around its center with a $6'000 rpm$ (rounds per minute). The angular velocity speed is $\omega = 2\pi\; 6'000/60 = 100\; 2\pi$. Because $E = \int\int_R \rho(r\omega)^2/2\, dxdy$, where $r = \sqrt{x^2+y^2}$, we have $E = \omega^2 I/2$, where $I = \rho \int\int_R (x^2+y^2)\, dxdy$ is the **moment of inertia**. For the square, $I = 4/3$. Its energy of the plate is $\omega^2 4/6 = 4\pi^2 100^2 4/6 Joule \sim 0.43 KWh$. (Run with it a 60 Watt bulb for 7 hours.)

WHERE CAN DOUBLE INTEGRALS OCCUR?

- Calculation of areas.
- Finding averages. Examples: average rain fall in US, average population in some area.
- Determining probabilities. Example: quantum probability.
- Moment of inertia $\int\int_R (x^2+y^2)\rho(x,y) dxdy$, center of mass $(\int\int_R x\rho(x,y)\, dxdy/M, \int\int_R y\rho(x,y)\, dxdy/M)$, with $M = \int\int_R dxdy$.
- Masses or volumes of cylindrical objects (height $\times$ ground area) $h\int\int_R 1\, dxdy$.
- Masses of cones (height $\times$ ground area)/3: $h\int\int_R 1 dxdy/3$.