Homework:

- pages 179-182, number 4a,c,10
- pages 191-193, numbers 4,8,10,12
- pages 199-200, numbers 2,6
- page 385, numbers 10,14

Suggested Problems:

- pages 192-193, 5,9,11

3D INTEGRATION. If $f(x,y,z)$ is a function of three variables and R is a region in space, then $\int\int\int_R f(x,y,z)\ dxdydz$ is defined as the limit of Riemann sum $\frac{1}{n^3}\sum_{\vec{x}_{ijk})\in R} f(\vec{x}_{ijk}$ for $n \to \infty$, where $\vec{x}_{ijk} = (\frac{i}{n}, \frac{j}{n}, \frac{k}{n})$.

In the same wasy as for double integrals, triple integrals can often be computed by **iterated integration**

TRIPLE INTEGRALS. As in two dimensions, we calculate a triple integral through iterated 1D integrals.

EXAMPLE. If R is the box $[0,1]\times[0,1]\times[0,1]$ and let $f(x,y,z) = x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dx\ dy\ dz.$$

CALCULATION. We start from the core $\int_0^1 24x^2y^3z\ dx = 12x^3y^3$, then integrate the middle layer: $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1.

WHAT DID WE DO? When we calculate the most inner integral, we fix z and y. The integral is then an average of $f(x,y,z)$ along a line intersected with the body. After completing the second integral, we have computed the average on the plane $z = const$ intersected with R. The most outer integral averages all these two dimensional sections.

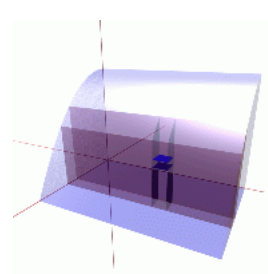
VOLUME OF THE SPHERE. (We will do this more elegantly later). The volume is

$$V = \int\int\int_R\ dxdydz = \int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz]dy]dx$$

After computing the inner integral, we have $V = 2\int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy]dx$.
To resolve the next layer, call $1 - x^2 = a^2$. The task is to find $\int_0^a \sqrt{a^2-x^2}\ dx$. Make the substitution $x/a = \sin(u), dx = a\cos(u)$ to write this as $a\int_0^{\arcsin(a/a)} \sqrt{1-\sin^2(u)} a\cos(u) du = a^2\int_0^{\pi/2}\cos^2(u)du = a^2\pi/2$. At this stage we have computed the area of a disc of radius a (which is the intersection of the sphere with the plane $x = const$. And we can finished up by calculating the last, the most outer integral

$$V = 2\pi/2 \int_{-1}^1 (1-x^2)\ dx = 4\pi/3\ .$$

TOTAL MASS. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$. The density of the body is 1.



$$\int_0^2 \int_0^6 \int_0^{4-x^2} dz\ dy\ dx = \int_0^2 \int_0^6 (4-x^2)\ dydx$$
$$= \ 6\int_0^2 (4-x^2)\ dx = 6(4x - x^3/3|_0^2 = 32$$

CENTER OF MASS. Compute the center of mass of the same body. The center of mass is $(24/32, 96/32, 256/180) = (3/4, 3, 8/5)$:

$$\int_0^2 \int_0^6 \int_0^{4-x^2} x\, dz\, dy\, dx = \int_0^2 \int_0^6 x(4-x^2)\, dydx = 6\int_0^2 x(4-x^2)\, dx = 24x^2/2 - 6x^4/4|_0^2 = 24$$

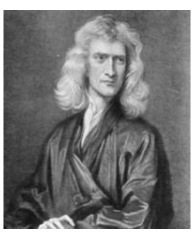
$$\int_0^2 \int_0^6 \int_0^{4-x^2} y\, dz\, dy\, dx = \int_0^2 \int_0^6 y(4-x^2)\, dydx = \int_0^2 18(4-x^2)\, dx = 18(4x - x^3/3)|_0^2 = 96$$

$$\int_0^2 \int_0^6 \int_0^{4-x^2} z\, dz\, dy\, dx = \int_0^2 \int_0^6 (4-x^2)^2/2\, dydx = 6\int_0^2 (4-x^2)^2/2\, dx = 3(16x - 8x^3/3 + x^5/5)|_0^2 = 256/5$$

SOME HISTORY OF COMPUTING VOLUMES. How did people come up calculating the volume $\int\int\int_R 1\, dxdydz$ of a body?

Archimedes ((-287)-(-212)): Archimedes's method of integration allowed him to find areas, volumes and surface areas in many cases. His method of exhaustion paths the numerical method of integration by Riemann sum. **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water.

Cavalieri (1598-1647): Cavalieri could determined area and volume using tricks like the **Cavalieri principle**. Example: to get the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibnitz(1646-1716): Newton and Leibnitz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it because possible to compute integrals using analytic tools.

MONTE CARLO COMPUTATIONS. Here is an other way to compute integrals: Suppose we want to calculate the volume of some body R inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** is to shoot randomly onto the unit cube and count the fraction of times, we hit. Let's see with Mathematica

$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of 10000 times. The volume is so measured as 0.5277. The volume of 1/8'th of the sphere is $\pi/6 = 0.524$

WHERE CAN TRIPLE INTEGRALS OCCUR?

- Calculation of volumes, masses.
- Finding averages. Examples: average algae concentration in a swimming pool.
- Determining probabilities. Example: quantum probability
- Moment of inertia $\int\int_R r(x,y,z)^2 \rho(x,y) dxdydz$, where $r(x,y,z)$ is the distance to the axes of rotation.
- Center of mass calculation $(\int\int\int_R x\, dxdydz/M, \int\int\int_R y\, dxdydz/M, \int\int\int_R z\, dxdydz/M)$.