Suggested Problems:

- pages 199-200, numbers 1,5a,7

- page 385 numbers 11,13,17

- page 207-209 numbers 1,5,7,9,11,15

INTEGRATION IN POLAR COORDINATES. Two dimensional integrals over circular regions are often better done using polar coordinates:

$$\int_0^r \int_0^{2\pi} f(r,\theta)\, r d\theta dr \ .$$

The only thing to remember is the factor r, whose origin is explained below.

EXAMPLE 1. Area of a disk of radius R

$$\int_0^R \int_0^{2\pi} r d\theta dr = 2\pi \frac{r^2}{2}|_0^R = R^2\pi \ .$$

EXAMPLE 2. Angular momentum of a disc of radius R

$$\int_0^R \int_0^{2\pi} r^2 r d\theta dr = 2\pi R^4/4 \ .$$

WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r,θ) plane is mapped to a sector segment in the (x,y) plane. It has approximately the area $r d\theta dr$. It is small for small r.

2. EXPLANATION. Look at the map $T : (r,\theta) \mapsto (r\cos(\theta, r\sin(\theta))$ which changes from Cartesian coordinates to polar coordinates. The Jacobean is $T' = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix}$ is a matrix with determinant r. This is a special case of a more general formula.

$$T : (r,\theta)$$
$$\mapsto$$
$$(r\cos(\theta), r\sin(\theta))$$
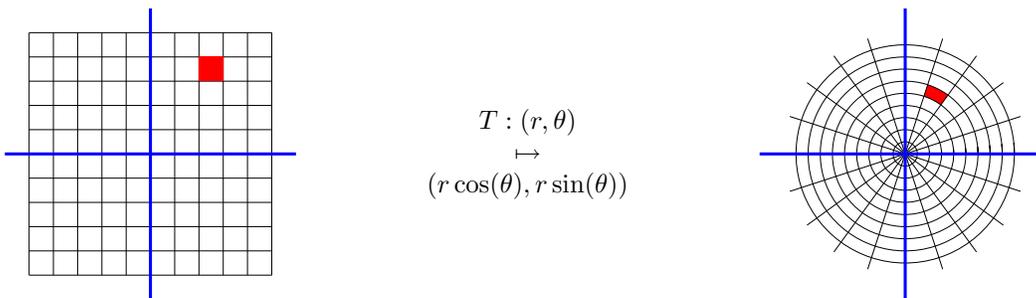
SUBSTITUTION IN HIGHER DIMENSIONS. If $(x,y) = T(u,v)$, the formula

$$\int \int_{T(R)} f(x,y)\, dxdy = \int \int_R f(u,v) \det(T')\, dudv$$

holds. The substitution formula is the same as in one dimensions, just take $\det(T')$ instead of T'. The polar coordinates $(x,y) = T(r,\theta) = (r\cos(\theta), r\sin(\phi))$ is an example.

1D CASE: $x = T(u), dx = T'(u)du$ for 1D definite integrals:

$$\int_{T(a)}^{T(b)} f(x)\, dx = \int_a^b f(T(u))T'(u)du .$$

Ex: $\int_{-1}^1 \sqrt{1-x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du$, $x = T(u) = \sin(u), dx = T'(u)du = \cos(u)du, f(T(u)) = \sqrt{1 - \sin^2(u)}$.

CYLINDRICAL COORDINATES. This are polar coordinates in the x-y plane and leaving the z coordinate. Take $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$. The integration factor $r = \det(T')$ is as in polar coordinates.

$$\int\int\int_{T(R)} f(x,y,z)\, dxdydz = \int\int\int_R f(r, \theta, z)\, rdrd\theta dz$$

COORDINATES OF CAMBRIDGE. On the website $http://cello.cs.uiuc.edu/cgi-bin/slamm/ip2ll/$ you can enter a host like $www.math.harvard.edu$ and get latitude and longitude of the host: $(lat, lon) = (42.365, -71.1)$. Using (r, θ, ϕ) coordinates, we obtain the position $(r, 90 - 42.365, -71.1)$ of the host in spherical coordinates. The site does not give the height, but we are about on see-level, so that $r = 6365km$.

EXAMPLE. Calculate the volume bounded by the parabolic $z = 1 - (x^2 + y^2)$ and the $x - y$ plane. In cylindrical coordinates, the paraboloid is $z(r, \phi) = 1 - r^2$:

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} rdzd\phi dr = \int_0^1 \int_0^{2\pi} (r - r^3)\, d\phi dr = 2\pi(r^2/2 - r^4/4)_0^1 = \pi .$$

SPHERICAL COORDINATES. Spherical coordinates use two angles, the longitude θ, the radius ρ as well as ϕ, the angle between the vector (x, y, z) and the z axes. Using these two angles and r, the coordinate change is
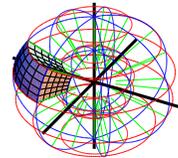
$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

The integration factor can be seen from the dimensions of a spherical wedge with dimensions $d\rho, \rho\sin(\phi)d\theta, \rho d\phi$.

$$\int\int\int_{T(R)} f(x,y,z)\, dxdydz = \int\int\int_R f(\rho, \theta, z)\, \rho^2\sin(\phi)\, d\rho d\theta d\phi$$
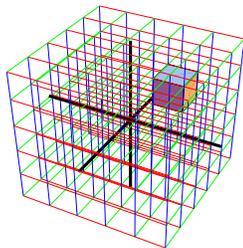
$$T : (\rho, \theta, \phi)$$
$$\mapsto$$
$$(\rho\cos(\theta)\sin(\phi),$$
$$\rho\sin(\theta)\sin(\phi),$$
$$\rho\cos(\phi))$$

VOLUME OF SPHERE. A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2\sin(\phi)\, d\phi d\theta d\rho .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi)d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\, d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\, d\rho = 4\pi R^3/3$.