Suggested Problems:

- pages 207-209 numbers 1,5,7,8,9,11,15
- Let V be the volume inside the cylinder where $0 \leq z \leq 10$ and $x^2 + y^2 \leq 1$. The density function for the interior of this cylinder is $\rho(x,y,z) = (10 - z^2)(1 - x^2 - y^2)$. Compute the total mass of the cylinder and compute the center of mass.

SUMMARY. Generalizing **substitution** in one dimensions $\int_{T(a)}^{T(b)} f(x)\ dx = \int_a^b f(T(u))T'(u)du$ like in $\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du$, $x = T(u) = \sin(u), dx = T'(u)du = \cos(u)du, f(T(u)) = \sqrt{1-\sin^2(u)}$, we have the change of variable formula

$$\int_{T(R)} f(\vec{x})\ d\vec{x} = \int_R f(T(\vec{u}))\ \det(T'(\vec{u}))\ du$$

if $\vec{x} = T(\vec{u})$. This is the same formula as in 1D, except of the determinant.

PURPOSE. While in one dimension, substitution is used primarily to **solve** integrals, in higher dimensions it serves also the purpose to **simplify the integration region**. Instead of integrating over a complicated figure $T(R)$ we can integrate over a cube R.

System	T	T'	$\det(T')$
DILATION	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$	abc
POLAR	$\begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$	$\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$	r
ZYLIND.	$\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \mapsto \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$	$\begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$	r
SPHER.	$\begin{bmatrix} r \\ \phi \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r\cos(\theta)\sin(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\phi) \end{bmatrix}$	$\begin{bmatrix} \cos(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) \\ \cos(\phi) & -r\sin(\phi) & 0 \end{bmatrix}$	$r^2\sin(\phi)$

T maps a the small cube $drd\theta d\phi$ into a segment of volume $dr(rd\phi)(r\sin(\phi)d\theta) = \sin(\phi)r^2 drd\phi d\theta$.

MASS-CENTER OF MASS. If a body R has mass density $\sigma(x,y,z)$, then $M = \int\int\int_R \sigma(x,y,z)dxdydz$ is the total mass of the body. The **center of mass** of R is the point

$$\int\int\int_R x\sigma(x,y,z)dxdydz/M, \int\int\int_R y\sigma(x,y,z)dxdydz/M, \int\int\int_R z\sigma(x,y,z)dxdydz/M)$$

MOMENT OF INERTIA. A body rotating around an axis L has the moment of inertia $I = \int\int\int_R \sigma(x,y,z)\rho(x,y,z)^2\ dxdydz$, where $\rho(x,y,z)$ is the distance to the axes of rotation and $\sigma(x,y,z)$ is the density of the body at the point (x,y,z). The moment of inertia can for example be used to calculate the **kinetic energy** of the body: $E = I\omega^2/2$. (Each little infinitesimal cube has mass $\sigma(x,y,z)dxdydz$ and moves with velocity $v(x,y,z) = r(x,y,z)\omega$, the kinetic energy is simply the "sum" over all the infinitesimal energies $dE(x,y,z) = \sigma(x,y,z)v^2(x,y,z)/2dxdydz$.)

EXAMPLE. Sphere of radius R and uniform density $\sigma = 1$. We use spherical coordinates, where the axes of rotation is the z-axes. The distance to the axes is $\rho(r,\theta,\phi) = r\sin(\phi)$. The integral is

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} r^4 \sin^3(\phi) d\phi d\theta dr = (4/3)2\pi R^5/5 = 2MR^2/5$$

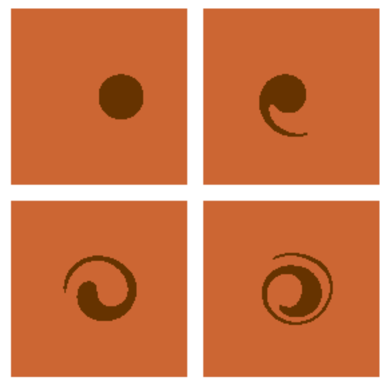
where $M = 4\pi R^3/3$ is the mass.
(**Note:** $\int_0^\pi \sin^3(\phi)\, d\phi = \int_0^\pi \sin(\phi)(1-\cos^2(\phi)\, d\phi = [\cos(\phi) - \cos(\phi)^3/3]_0^\pi = 2 - 2/3 = 4/3$.)

EARTH. $M = 5.97210^{24}kg$. $R = 6378km = 6.37810^6 m$. $I = 9.71710^{37}kgm^2$. Angular velocity speed $\omega = 2\pi/86400s^{-1}$. The rotational kinetic energy of the earth is $E = I\omega^2/2 = 2.5710^{29}kgm^2/s^2(Joule)$. The maximal amount of power, a human can generate **sustainably** is P=125 Watt (1W=1J/s). How long would you have to pedal to rotate the earth from rest to the current rotation? $E/P = 2.5710^{29}/125J/(J/s) = 2.0610^{27}s$. The universe is $15MiaYears = 15 * 10^9 Years = 1.62910^{17}s$ old. So you would have to pedal about 100 billion times the age of the universe to finish with this Sysiphus job. Believe me, you would look **really old** after that.

MOMENTS OF INERTIA OF SOME BODIES.

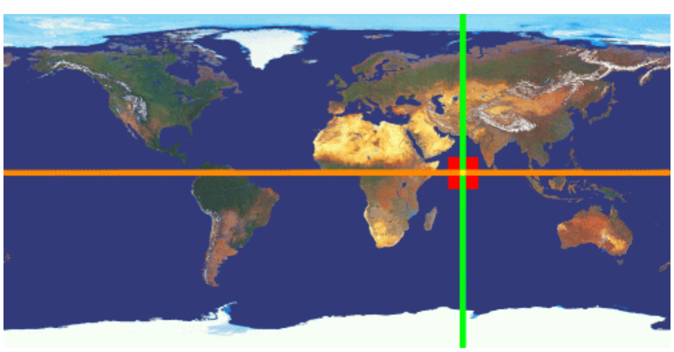
Hollow sphere	arbitrary axes	$I = (2/3)MR^2$
Full sphere	arbitrary axes	$I = (2/5)MR^2$
Hollow cylinder	axes of rotation=axes of cylinder	$I = MR^2$
Full cylinder	axes of rotation=axes of cylinder	$I = MR^2/2$
Full cylinder of length L	axes orthogonal to cylinder axes	$I = ML^2/12$
Rectangular plane width=a, length=b	axes of rotation orthogonal to plane	$I = M(a^2+b^2)/12$

CHOCOLATE QUIZ. You have made a nice bowl of chocolate pudding and add a drop of chocolate $R = \{x^2 + y^2 \leq 1/4\}$. The mixing of the pudding is done as follows. You apply alternatively a stirring transformation $T_P : (r,\theta) \mapsto (r, \theta + (1-r)\theta)$ for $r \leq 1$ in polar coordinates around a point P, then the same transformation T_Q centered at an second point Q. Calculate the area of the chocolate after 3 such stirs.

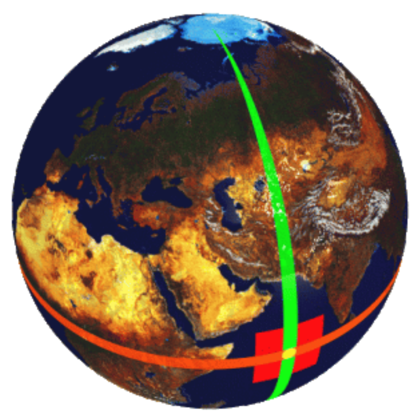


ANSWER. Each transformation can be written as compositions of translations and maps $T(x,y) = (x\cos(f(r)) - y\sin(f(r)), x\sin(f(r)) + y\cos(f(r)))$ is area preserving: the Jacobean $T'(x,y) = f'(r)\begin{bmatrix} \cos(f(r)) & -\sin(f(r)) \\ \sin(f(r)) & \cos(f(r)) \end{bmatrix}$ has determinant 1. Therefore $\int\int_{T(R)} dxdy = \int\int_R 1 dxdy$, the volumes are the same.

LATITUDE AND LONGITUDE. Maps of the earth are often given by a map T mapping a rectangle $[0,2\pi]\times[0,\pi]$ to the sphere.



T
$\rightarrow$



The (θ,ϕ) coordinates are familiar to you. A map of the earth usually shows the rectangle R which is mapped to the surface of the earth using $(\theta,\phi) \mapsto (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi)$. The north upper boundary of the rectangle is mapped to the north pole, the lower boundary of the rectangle is mapped to the south pole.

You see on the texture map to the left how the area near the south and north poles are stretched. The determinant $r^2\sin(\phi)$ of the map T corrects this. If we want to calculate the area of the ice region we integrate on the rectangle but have there to take the density $r^2\sin(\phi)$.