Homework:

- pages 207-209 number 2, (no technology), 10,12
- pages 273-275 numbers 2,6,8,10
- pages 279-280 numbers 2,4
- Use polar coordinates to integrate the function $\log(x^2+y^2)$ over the region where $x^2+y^2 \leq 1$. (Hint: An integration by parts can help with the evaluation of the integral.)
- Use spherical coordinates to integrate the function $1/\sqrt{x^2+y^2+z^2}$ over the region, where $x^2+y^2+z^2 \leq 1$.

Suggested Problems: pages 273-275, numbers 1,5,7,9,11

LINE INTEGRALS (RECALL). If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field and $\gamma : t \mapsto r(t) = (x(t), y(t)), t \in [a,b]$ is a cuve, then

$$\int_\gamma F \; ds = \int_a^b F(x(t), y(t)) \cdot (x'(t), y'(t)) \; dt$$

is called the **lineintegral of F along γ**.

THE CURL OF A 2D VECTOR FIELD. The curl of a 2D vector field $F(x,y) = (P(x,y), Q(x,y))$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \; .$$

INTERPRETATION. curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times F = \text{curl}(F)$ for the curl of F because the cross product of (∂_x, ∂_y) with F is the expression. (In three dimensions, $\text{curl}(F) = \nabla \times F$ is a vector field again.)

EXAMPLES.
1) $F(x,y) = (-y, x)$. $\text{curl}(F)(x,y) = 2$.
2) $F(x,y) = \nabla U$, (conservative field = gradient field = potential) Because $P(x,y) = U_x(x,y), Q(x,y) = U_y(x,y)$, we have $Q_x - P_y = U_{yx} - U_{xy} = 0$.

GREEN'S THEOREM. (1827) If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field in the plane and R is a region in the plane which has as a boundary a piecewise smooth closed curve γ traversed in the direction so that the region R is "to the left". Then

$$\int_\gamma F \cdot ds = \int \int_R \text{curl}(F) \; dxdy$$

Note that for a region with holes, the boundary consists of many curves and that they are traversed always to that R is to the left.

GEORGE GREEN (1793-1841) was one of the most remarkable of nineteenth century physicists, a self-taught mathematician whose work has contributed greatly to modern physics. Unfortunately, there seems to be no picture of Green. George Green was the sun of a baker. Her worked at his father's mill and studied math and physics in his spare time. Greens theorem has been found independently by Michael Ostrogradsky (1801-1862) in 1928, the same year, Green published a memoir "An essay on the application of mathematical analysis to the theory of electricity and magnetism" containing the result.

GENERAL FUNDAMENTAL THEOREM OF CALCULUS (Preview). Green's theorem is a general theorem $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are d such theorems in dimensions d. In dimension 2, Green's theorem is the second (the first is the fundamental theorem of line integrals). In three dimensions, there will be two more theorems (Stokes and Gauss, see later).

dim	dim(R)	theorem
1D	1	Fund. thm of calculus
2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem (* later *)
3D	3	Gauss theorem (* later *)

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl (* later *)
$3 \mapsto 1$	$\nabla \cdot F$	divergence (* later *)

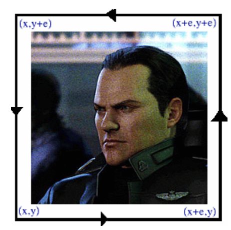
As a physisist, you will deal also corresponding theorems in 4 dimensions. In 4D, there are 4 theorems. A uniform language called **calculus of differential forms** allows to generalize Green to arbitrary dimensions.

EXAMPLE. If F is a gradient field, then both sides are zero:

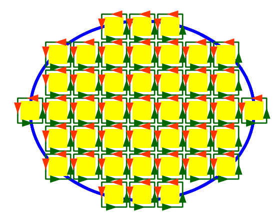
$\int_\gamma F \cdot ds$ vanishes by the fundamental theorem for line integrals.	$\int_R \text{curl}(F) \cdot ds$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(U)) = 0$.

The fact that $\text{curlgrad}(U) = 0$ can also be seen from $\nabla \times \nabla U$ and the general fact that the cross product of two identical vectors is 0. Just treat ∇ as a vector.

WHERE IS THE PROOF? (Quote: General Hein in "Final Fantasy"). To prove Green's theorem look first at a small square $R = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $F = (P,Q)$ along the boundary is $\int_0^\epsilon P(x+t, y)dt + \int_0^\epsilon Q(x+\epsilon, y+t)\, dt - \int_0^\epsilon P(x+t, y+\epsilon)\, dt - \int_0^\epsilon Q(x, y+t)\, dt$. (Note also that this line integral measures the "circulation" at the place (x,y).)



Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, the line integral is $(Q_x - P_x)\epsilon^2$ is about the same as $\int_0^\epsilon \int_0^\epsilon \text{curl}(F)\; dxdy$. All identities are true in the limit $\epsilon \to 0$.
To prove the statement for a general region R, chop it up into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.

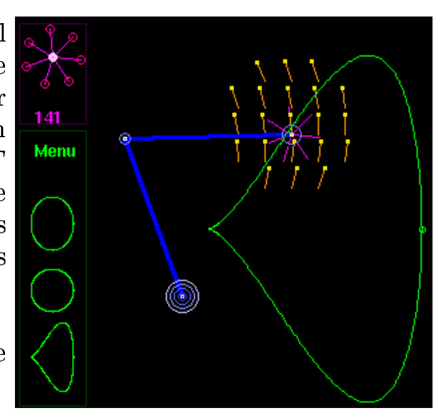


APPLICATION: AREA FORMULAS. The vector field $F(x,y) = (P,Q) = (-y, 0)$ has vorticity $\text{curl}(F(x,y)) = 1$. The right hand side in Green's theorem is the **area** of R.

EXAMPLE. Let R be the region under the graph of a function $f(x)$ on $[a,b]$. The lineintegral around the boundary of R is 0 from $(a,0)$ to $(b,0)$ because $F(x,y) = 0$ there. The lineintegral is also zero from $(b,0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a,0)$ because $Q = 0$. The lineintral on $(t, f(t))$ is $-\int_a^b ((-y(t), 0) \cdot (1, f'(t))dt = \int_a^b f(t)\; dt$. Green's theorem assures that this is the area of the region below the graph.

APPLICATION. THE PLANIMETER. The planimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in ingeneering to measure the size of profiles. There is a vector field F associated to a planimeter (put a vector of length 1 orthogonally to the arm). One can prove that F has vorticity 1. The planimeter calculates the line integral of F along a given curve. Green's theorem assures it is the area.

The picture to the right shows a Java applet which allows to explore the planimeter (from a CCP module by O. Knill and D. Winter, 2001).



To explore the planimeter, visit the URL http://ncd3.math.harvard.edu/ccp/materials/mvcalc/green/index.html