11/7/2001, SURFACES (Section 5.4) — Math 21a, O. Knill

Suggested Problems: pages 279-280-275, number 1, no technology

PARAMETRIZED SURFACES. The image of a map $X(u,v) = (x(u,v), y(u,v), z(u,v))$ is a **surface**. X is called the parameterization of the surface. It is given by three functions $x(u,v), y(u,v), z(u,v)$ of two variables. In Mathematica, you have seen this as "ParametricPlot3D".

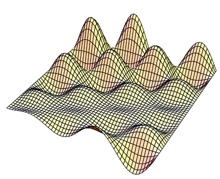
THREE WAYS TO REPRESENT A SURFACE.

- Graphs of a function $f(x,y)$
- Solutions of an equation $f(x,y,z) = 0$
- parameterizations as image of $X(u,v)$.

EXAMPLE. Sphere. The upper half sphere is the graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$, the sphere is the set of (x,y,z) for which $x^2+y^2+z^2 = 1$. The sphere is also the image of the parametrisation $X(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$. From all these representations, the parametric representation is the most flexible one.

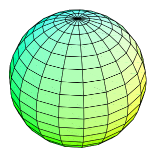
NOTE. There are surfaces, which can also not represented as an image of a single parametrisation X. In general, one needs different patches. This is made precise with the notion of a manifold.

EXAMPLES of parameterized surface $(u,v) \mapsto X(u,v) = (x(u,v), y(u,v), z(u,v))$



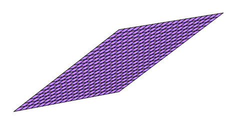
Graphs

$$\begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$



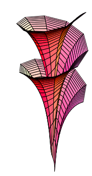
Sphere

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$$



Plane through points P, Q, R

$$P + u(Q-P) + v(R-P)$$



Dini surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix}$$

NORMAL VECTOR. If we fix u, then $v \mapsto X(u,v)$ is a curve on the surface. Its velocity vector $X_v(u,v)$ is tangent to the surface. Similarly, also $X_u(u,v)$ is tangent to the surface. The vector $X_u \times X_v$ is therefore normal to the surface at the point $X(u,v)$.

EXAMPLE. The sphere $X(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$.
$X_u(u,v) = (-\sin(u)\sin(v), \cos(u)\sin(v), 0)$. $X_v(u,v) = (\cos(u)\cos(v), \sin(u)\cos(v), -\sin(v))$. $n = X_u \times X_v = (-\cos(u)\sin^2(v), -\sin(u)\sin^2(v), -\cos(v)\sin(v)) = -\sin(v)X(u,v)$. The normal vector is parallel to the vector $X(u,v)$ as it should indeed be for the sphere.

TANGENT PLANE. Having the normal, we also know the tangent plane at a point $(x_0,y_0,z_0) = X(u,v)$. If $(a,b,c) = X_u \times X_v$, then $ax+by+cz = d$ is the tangent plane, where d is obtained as $d = ax_0+by_0+cz_0$.

EXAMPLE. Calculate the tangent plane at $X(\pi/2,\pi/2) = (0,1,0)$ for the sphere. We know that $X(u,v)$ is normal to the surface at the point $X(u,v)$. Therefore $(a,b,c) = (\cos(\pi/2)\sin(\pi/2), \sin(\pi/2)\sin(\pi/2), \cos(\pi/2) = (0,1,0)$ is the normal vector at $X(\pi/2,\pi/2) = (0,1,0)$. The equation for the plane is $ax+by+cz = d$ which is in this case $y = 1$.

THE DERIVATIVE. The derivative of $X : \mathbf{R}^2 \to \mathbf{R}^3$ is the 3x2 matrix $\begin{bmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \\ z_u(u,v) & z_v(u,v) \end{bmatrix} =$ $\begin{bmatrix} \nabla x(u,v) \\ \nabla y(u,v) \\ \nabla z(u,v) \end{bmatrix} = \begin{bmatrix} | & | \\ X_u & X_v \\ | & | \end{bmatrix}$. The colums are the tangent vectors X_u, X_v.

EXPANSION RATE. If we look at a small square in the (u, v)-plane with area ϵ^2, then $|X_u \times X_v|\epsilon^2$ is the area of the image.

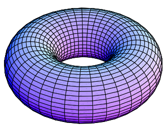
EXAMPLE. In the case of the unit sphere, we have seen that the expansion rate is $|\sin(v)|$. In the case of a sphere of radius r it would be $r^2 \sin(v)$.

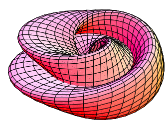
AREA.

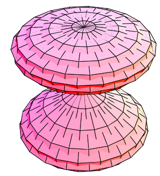
$$\int \int_R |X_u(u,v) \times X_v(u,v)| \, dudv$$

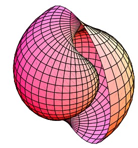
is the area of the surface. We will come back to this next time.

MORE EXAMPLES. Parameterized surfaces $(u, v) \mapsto X(u, v)$









Torus

$$\begin{bmatrix} (2+\cos(v))\cos(u) \\ (2+\cos(v))\sin(u) \\ \sin(v) \end{bmatrix}$$

Klein bottle

$$\begin{bmatrix} (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\cos(2u) \\ (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\sin(2u) \\ \frac{3}{4}\sin(u)\sin(2v)+\cos(u)\sin(4v) \end{bmatrix}$$

Eight surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v)+\log(\tan(\frac{v}{2}))+\frac{u}{5} \end{bmatrix}$$

Snail surface

$$\begin{bmatrix} \cos(u)\sin(2v) \\ \sin(u)\sin(2v) \\ \sin(v) \end{bmatrix}$$

WHERE DO SURFACES OCCUR?

- **Computer graphics**. i.e. modeling of faces, terrains, cars, spaceships etc.
- **Level surfaces**. Level surfaces of functions of three variables. Example, surface of constant temperature in the ocean.
- **Graphs**. For example: height functions, probability distribution functions.
- **Intuition** Intuition for higher dimensional surfaces (or manifolds) is often obtained from 2-dimensional surfaces in three dimensional space. Higher dimensional surfaces appear everywhere. Our universe for example is modeled as a four dimensional manifold in general relativity. The planetary motion in our the solar system is modeled by a flow on a $9 * 6 - 10$ dimensional surface.
- **D-branes**. In super-string theory, surfaces called **Dp-branes** appear. (The letter D stands for "Dirichlet"). $D1$-branes are called D-strings.
- **Artistics**. Last but not least, there is the artistic and esthetic aspect to surfaces.

The entrance of the science center shows a beautyful surface (see photo). You walk by this surface every day. The website $http : //www.courses.fas.harvard.edu/\tilde{m}ath21a/labs/selection2$ shows some of your creations of surfaces.