HOMEWORK. pgs. 273-275 numbers 2,8,10, pgs 279-280, numbers 2 4, pgs 286-287 number 2 (skip plotting task), 4,6, pgs 292-293 number 2,4. Also do:
1) Take a parameterized curve $\gamma : (f(u), g(u))$, where $g(u) > 0$ in the $x-y$-plane and revolve it about the x-axis in space to create a surface (called a "surface of revolution").
a) Show that $X(u, v) = (f(u), g(u)\cos(v), g(u)\sin(v))$ parameterizes the surface, where $v \in [0, 2\pi)$.
b) Sketch the surface of revolution if $f(u) = u, g(u) = 1$ and in the case $f(u) = \cos(u), g(u) = \sin(u) + 2$.
2) Find a parameterization for the surface obtained by revolving the curve $y = e^x$ about the x-axis.
3) Find a parameterization for the surface, where $x - y^2 + z^4y^4 = 0$.
4) Find a parameterization for the $y > 0$ part of the surface, where $y^2 - ((x^4 + z^4)^8 + 1) = 0$.
5) Find the area of the part of the plane $x + y + z = 1$, where $x, y, z \in [0, 1]$.
6) Integrate the function $f(x, y, z) = x - y$ over the part of the plane $x + 2y + 3z = 6$, where $x, y, z \in [0, 1]$.

MIDTERM TOPICS.

- Extrema of functions of several variables
- Extrema of functions defined on surfaces.
- Linear approximation. Level curves.
- Integration in two variables

LEVELS OF UNDERSTANDING.

1. I) OBJECTS. Know **what** are the objects, definitions, theorems, names, jargon and history.
2. II) TASKS. Know **how** to work with the objects, run algorithms.
3. III) UNDERSTANDING. See different aspects, contexts. **Why** is it done like this?
4. IV) APPLY. Extend the theory, apply to new situations, invent new objects, ask question: **Why not...?**

I) Definitions and objects.

CONSTRAINED EXTREMUM of F constrained by $G = c$ are obtained where $\nabla F = \lambda \nabla G, G = c$.
CHAIN RULE. If $r(t) = (x(t), y(t))$ is a curve and $F(x, y)$ is a function, then $d/dt F(r(t)) = \nabla F(r(t)) \cdot r'(t)$.
CRITICAL POINT. $\nabla F(x, y) = (0, 0)$. Is also called **stationary point**.
DOUBLE INTEGRAL. $\int_a^b \int_{f(x)}^{g(x)} f(x, y)\, dydx$ is an example of a double integral.
GRADIENT. $F(x, y)$ function of two variables, $\nabla F(x, y) = (\partial_x F(x, y), \partial_y F(x, y)) = (F_x(x, y), F_y(x, y))$.
HESSIAN. $F(x, y)$ function of two variables. The Hessian is the matrix $H(x, y) = \begin{bmatrix} F_{xx}(x, y) & F_{xy}(x, y) \\ F_{yx}(x, y) & F_{yy}(x, y) \end{bmatrix}$.
LEVEL SURFACE. $F(x, y, z) = 0$ has gradients $\nabla F(x, y, z)$ as normals.
LINEAR APPROXIMATION. $L(x, y) = F(x_0, y_0) + \nabla F(x_0, y_0) \cdot (x - x_0, y - y_0)$.
LOCAL MAXIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) < 0$ is a local maximum.
LOCAL MINIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) > 0$ is a local minimum.
QUADRATIC APPROXIMATION. $Q(x, y) = L(x, y) + [H(x, y)(x - x_0, y - y_0)] \cdot (x - x_0, y - y_0)/2$.
SADDLE POINT. A critical point for which $\det(H(x, y)) < 0$.
TRIPLE INTEGRAL. Example: $\int_a^b \int_{f(x)}^{g(x)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)\, dzdydx$.

II) Algorithms

INTEGRATION OVER A DOMAIN R.
1) Eventually chop the region into pieces which can be parametrized.
2) Start with one variable, say x and find the smallest x-interval $[a, b]$ which contains R.
3) For fixed x, intersect the line $x = const$ with R to determine the y-bounds $[f(x), g(x)]$.
4) Evaluate the integral $\int_a^b \left[\int_{f(x)}^{g(x)} F(x, y)\right] dydx$.
5) Solve the double integral by 1D integration starting from inside.
6) In case of problems with the integral, try to switch the order of integration. (Go to 2) and start with y).

EXAMPLE. Integrate x^2y^2 over the triangle $x + y/2 \le 3, x > 0, y > 1$. The triangle is contained in the strip $0 \le x \le 3$. The x-integration ranges over the interval $[0, 3]$. For fixed x, we have $y \ge 1$ and $y \le 2(3 - x)$ which means that the y bounds are $[0, 2(3 - x)]$. The double integral is $\int_0^3 \int_1^{6-2x} x^2y^2\, dydx$.

FINDING THE MAXIMUM OF A FUNCTION F(x,y) OVER A DOMAIN R with boundary G(x,y)=c.

1) First look for all stationary points $\nabla F(x,y)$ in the interior of R.
2) Eventually classify the points in the interior by looking at $\det(H)(x,y)$, $H_{xx}(x,y)$ at the critical points.
3) Locate the critical points at the boundary by solving $\nabla F(x,y) = \lambda \nabla G(x,y), G(x,y) = c$.
4) List the values of F evaluated at all the points found in 1) and 3) and compare them.

EXAMPLE. Find the maximum of $F(x,y) = x^2 - y^2 - x^4 - y^4$ on the domain $x^4 + y^4 \leq 1$. $\nabla F(x,y) = (2x - 4x^3, -2y - 4y^3)$. The critical points inside the domain are obtained by solving $2x - 4x^3 = 0, -2y - 4y^3 = 0$ which means $x = 0, x = 1/4$, $y = 0, y = -1/4$. We have four points $P_1 = (0,0), P_2 = (1/\sqrt{2}, 0), P_3 = (0, -1/\sqrt{2}), P_4 = (1/\sqrt{2}, -1/\sqrt{2})$.
The critical points on the boundary are obtained by solving the Lagrange equations $(2x - 4x^3, -2y - 4y^3) = \lambda(4x^3, 4y^3), x^4 + y^4 = 1$. Solutions (see below) are $P_5 = (0,-1), P_6 = (0,1), P_7 = (-1,0), P_8 = (1,0)$. A list of function values $F(P_1) = 0, F(P_2) = 1/2 - 1/4, F(P_3) = -1/2 - 1/4, F(P_4) = -2/4, F(P_5) = -2, F(P_6) = -2, F(P_7) = 0, F(P_8) = 0$ shows that P_2 in the interior is the maximum. Indeed, the Hessian at this point is $H = \text{diag}(-1,-2)$ which has positive determinant and negative H_{11}.

SOLVING THE LAGRANGE EQUATIONS.
1) Write down the equations neatly.
2) See whether some variable can be eliminated easily. λ can always be eliminated.
3) If some variable can be eliminated easily, go back to 1) using one variable less and repeat.
4) Try to combine, rearrange, simplify the equations. The system might not have an algebraic solution.
5) Finally, (if possible) check with a symbolic manipulation program, whether you overlooked something.

EXAMPLE.

$$\begin{aligned} 2x - 4x^3 &= 4\lambda x^3 \\ -2y - 4y^3 &= 4\lambda y^3 \\ x^4 + y^4 &= 1 \end{aligned}$$

$$\begin{aligned} 2x - (4+4\lambda)x^3 &= 0 \\ -2y - (4+4\lambda)y^3 &= 0 \\ x^4 + y^4 &= 1 \end{aligned}$$

$$\begin{aligned} x = 0 \quad &\text{or} \quad x = 1/\sqrt{2+2\lambda} \\ y = 0 \quad &\text{or} \quad y = -1/\sqrt{2+2\lambda} \\ x^4 + y^4 &= 1 \end{aligned}$$

If x=0, then y=1, or y=-1, if $y = 0$ then $x = 1$ or $x = -1$. There are 4 critical points.

III) "Understanding". Try to answer questions like:

- What do the Lagrange equations mean geometrically? Explain the method to somebody with a drawing.
- Explain why only critical points can be candidates for maxima or minima of a function $F(x,y)$.
- What can happen at a critical point if the quadratic approximation at this point vanishes?
- What is the geometric meaning of the entry H_{xy} in the Hessian?
- Discuss the chain rule $F(r(t))$ for $F(x,y) = \sqrt{x^2}$, $r(t) = (x(t), y(t)) = (t, t^2)$.
- In which situations is the Lagrange multiplyer $\lambda = 0$?
- If a function $F(x,y)$ is replaced by its quadratic approximation $Q(x,y)$, what do you expect the error $F(0.01, 0.01) - Q(0.01, 0.01)$ to be?
- Can you figure out an example of a function, where the linear approximation works, but where the quadratic approximation fails?
- Can we have a circular shaped island on which all critical points are saddles?
- Find a criterium for maximum or minimum for functions $F(x,y,z)$ of 3 or 4 variables.

IV) Apply to new situations.

Problem solving and creativity skills are acquired best by "doing" it, by pondering over new questions, working on specific problems. Nevertheless, there is also theoretical help: for example, G. Polya (1887-1985)'s book "how to solve it" gives some general advise on "how to solve problems". Abstract: "How to solve a problem?":

I) Understand the problem.
II) Think of a plan by solving subproblems. Connect with older problems.
III) Walk along the plan while controlling each step.
IV) Check the result. Is the result obvious? Is the method useful for other problems?