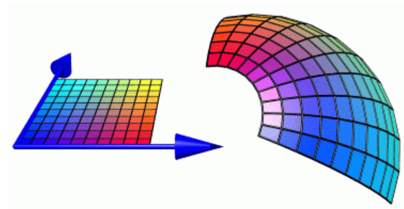
Suggested Problems:

1. 286-287, number 1,3
2. The charge density on the surface of the infinitely long cylinder $x^2+y^2 \leq 1$ is given by $\sigma(x,y,z) = \exp^{-|z|}$. Compute the total charge on the cylinder.
3. The charge density on the sphere $x^2+y^2+z^2 = 1$ is given by $\sigma(x,y,z) = z^2$. Compute the total charge on the sphere.



INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If S is a surface, then $\int\int_S f(x,y)\ dS$ should be an average of f on the surface. If $f(x,y) = 1$, then $\int\int_S\ dS$ should be the area of the surface. If S is the image of X under the map $(u,v) \mapsto X(u,v)$, then $dS = |X_u \times X_v| dudv$.

DEFINITION. Given a surface $S = X(R)$, where R is a domain in the plane and where $X(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface integral of $f(u,v)$ on S is defined as

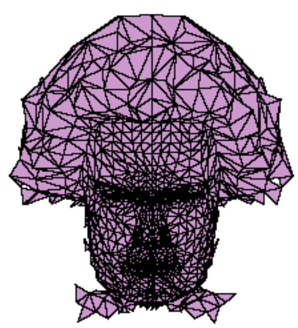
$$\int\int_S f\ dS = \int\int_R f(u,v)|X_u \times X_v|\ dudv\ .$$

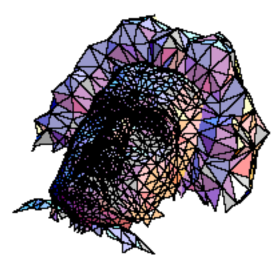
INTERPRETATION. If $f(x,y)$ measures a quantity then $\int\int_S f\ dS$ is the average of the function f on S. Examples:
-if $f(u,v)$ is **amount of red color** at the spot $X(u,v)$ of the surface, then $\int\int_S f\ dS$ is the total average value of red on the surface.
-If $f(u,v)$ is the charge density at the point $X(u,v)$, then $\int\int_S f\ dS$ is the **total charge** on the surface S.
- If $f(x,y) = 1$, then $\int\int_S f\ dS$ is the **area** of S.

EXPLANATION OF $|X_u \times X_v|$. The vector X_u is a tangent vector to the curve $u \mapsto X(u,v)$, when v is fixed and the vector X_v is a tangent vector to the curve $v \mapsto X(u,v)$, when u is fixed. The two vectors span a parallelogram with area $|X_u \times X_v|$. A little rectangle spanned by $[u, u+du]$ and $[v, v+dv]$ is mapped by X to a parallelogram spanned by $[X, X+X_u]$ and $[X, X+X_v]$.

A simple case: consider $X(u,v) = (2u, 3v, 0)$. This surface is part of the x-y plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $X_u \times X_v = (0,0,6)$ and we see for example that the area of $X(R)$ is 6 times the area of R.

UV-MAPPINGS. In computer graphics, the map X is called a **u-v map**. Such maps are used to apply textures to three-dimensional models. There are software utilities, which allows an artist to draw directly onto the R domain. The software then applies the map automatically onto the surface. You can paint like this on surfaces in space.

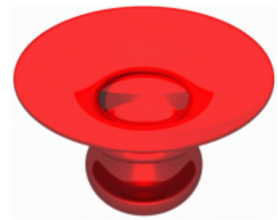




THE AREA OF THE SPHERE. The map $X = (u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ maps the rectangle $R: [0, 2\pi] \times [0, \pi]$ onto the sphere. We have calculated $X_u \times X_v$ already. It was $\sin(v) X(u,v)$. So, $|X_u \times X_v| = |\sin(v)|$ and $\int\int_R 1 dS = \int_0^{2\pi} \int_0^{\pi} \sin(v)\ dvdu = 4\pi$.

GRAPHS. For surfaces $(u,v) \mapsto (u,v,f(u,v))$, we have $X_u = (1,0,f_u(u,v))$ and $X_v = (0,1,f_v(u,v))$. The cross product $X_u \times X_v = (-f_u,-f_v,1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region R is $\int\int_R \sqrt{1+f_u^2+f_v^2}$.
EXAMPLE. The surface area of the paraboloid $z = f(x,y) = x^2+y^2$ is (use polar coordinates) $\int_0^{2\pi} \sqrt{1+4r^2} r \, dr d\theta = 2\pi(2/3)(1+4r^2)^{3/2}/8|_0^1 = \pi(5^{3/2}-1)/6$.

SURFACES OF REVOLUTION. (see also homework) Consider a positive function $f(x)$ on an interval $[a,b]$ on the x axes and and rotate the graph around the x-axes. This surface is parameterized by $(u,v) \mapsto X(u,v) = (v, f(v)\cos(u), f(v)\sin(u))$ on $R = [0,\pi] \times [a,b]$ and is called a **surface of revolution**. We have $X_u = (0,-f(v)\sin(u), f(v)\cos(u))$, $X_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $X_u \times X_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$ which has the length $|X_u \times X_v| = |f(v)|\sqrt{1+f'(v)^2} \, dudv$.

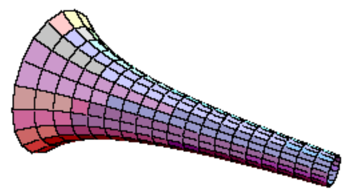


EXAMPLE. Consider $f(x) = x$ on the interval $[0,1]$. The area of this surface (a cone) is $\int_0^{2\pi} \int_0^1 x\sqrt{1+1} \, dvdu = 2\pi\sqrt{2}/2 = \pi\sqrt{2}$.
P.S. In computer graphics, surfaces of revolutions are constructed from a few prescribed points $(x_i, f(x_i))$. The machine constructs a function (**spline**) and rotates this around the x-axes (**lathe**).

GABRIEL'S HORN. Take $f(x) = 1/x$ on the interval $[1,\infty)$.

Volume: The volume is (use cylindrical coordinates in the x direction) $\int_1^\infty \pi f(x)^2 \, dx = \pi \int_1^\infty 1/x^2 \, dx = \pi$.

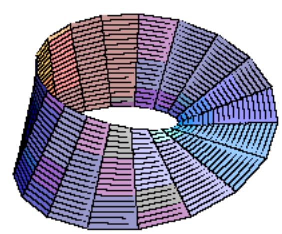
Area: The area is $\int_0^{2\pi} \int_1^\infty 1/x\sqrt{1+1/x^4} \, dx \geq 2\pi \int_1^\infty 1/x \, dx = 2\pi \log(x)|_0^\infty = \infty$.



The Gabriels horn is a surface of finite volume but with infinite surface area! You can fill the horn with a finite amount of paint, but this paint does not suffice to cover the surface of the horn!

Question. How long does a Gabriel horn have to be so that its surface is $500cm^2$ (area of sheet of paper)? Because $1 \leq \sqrt{1+1/x^4} \leq \sqrt{2}$, the area for a horn of length L is between $2\pi \int_1^L 1/x \, dx = 2\pi\log(L)$ and $\sqrt{2}2\pi\log(L)$. In our case, L is between $e^{500/(\sqrt{2}2\pi)} \sim 2 * 10^{24} cm$ and $e^{500/(2\pi)} \sim 4 * 10^{34} cm$. Note that the universe is about 10^{26} cm long! It could hardly accomodate a Gabriel horn in our universe if it should have the surface area of a sheet of paper!

MÖBIUS STRIP. The surface $X(u,v) = (2 + v\cos(u/2)\cos(u), (2+v\cos(u/2))\sin(u), v\sin(u/2))$ parametrized by $R = [0,2\pi] \times [-1,1]$ is called a **Möbius strip**.
The calculation of $X_u \times X_v$ is a bit messy and gives (no shame to use technology) $|X_u \times X_v| = 4 + (3v^2)/4 + 4v\cos(u/2) + (v^2 * \cos(u))/2$. The integral over $[0,2\pi] \times [-1,1]$ is 17π.



TRICK QUESTION. If we build the Moebius strip with a piece of paper. Suppose the density is 1/1000. What is the relation between the area of the surface and the weight of the surface?
You could argue that the surface area multiplied with the density gives twice the weight because the Möbius strip is a surface with a single side and we cover both sides of the surface. Is this true?

P.S. You can admire an OpenGL implementation of an Escher theme using the Moebius strip with "xlock -inwindow -mode moebius" on an x-terminal. P.P.S. A patent was once assigned to the idea to use a Moebius strip as a **conveyor belt**. It would last twice as long as an ordinary one.