

Suggested Problems: page 292-293 numbers 1,3,7,9

INTEGRAL OF A VECTOR FIELD ON A SURFACE. If S is a surface, and F is a vector field in space, the integral $\int_S F \cdot dS$ is called the **flux** of F through the surface S .



DEFINITION. Given a surface $S = X(R)$, where R is a domain in the plane and where $X(u, v) = (x(u, v), y(u, v), z(u, v))$. The **flux integral** of F through S is defined as

$$\int_S F \cdot dS = \int_R \int F(X(u, v)) \cdot (X_u \times X_v) \, dudv.$$

INTERPRETATION. If F = fluid velocity field, then $\int_S F \cdot dS$ is the flux of fluid passing through S .

EXAMPLE. Let $F(x, y, z) = (0, 1, z^2)$ and let S be the sphere with $X(u, v) = (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v))$, $X_u \times X_v = \sin(v)X$ so that $F(X(u, v)) = (0, 1, \cos^2(v))$ and $\int_0^{2\pi} \int_0^\pi \sin(v)(0, 1, \cos^2(v)) \cdot (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \, dudv$. The integral is $\int_0^{2\pi} \int_0^\pi \sin^2(v) \sin(u) + \cos^3(v) \sin(v) \, dudv = 0$.

Look at the vector field. Most flux passes in at the south pole, most flux passes out at the north pole. However, there is some symmetry and what enters in the south leaves in the north. The total flux is zero.

WHAT IS THE FLUX INTEGRAL? Because $n = X_u \times X_v / |X_u \times X_v|$ is a unit vector normal to the surface and on the surface, $F \cdot n$ is the normal component of the vector field with respect to the surface, we can write $\int_S F \cdot dS = \int_S F \cdot n \, dS$.

The flux integral is the surface integral of the function $f(u, v) = (F(X(u, v)) \cdot n(X(u, v)))$.

BIKING IN THE RAIN. As a matter of fact, during rain you get soaked more on a bike than when you walk foot. The Reason? For a biker who drives with speed V along the x axes, the rain is a fluid with velocity $F = (-V, 0, W)$, where W is the speed of the rain drops falling along the z -axes. If the biker is a rectangle $[0, 1/2] \times [0, 1]$ parametrized by $X(u, v) = (0, u, v)$ in the $y - z$ plane, the amount of water soaked up in unit time by the clothes is the flux of the rain through that surface. We know $X_u \times X_v = (1, 0, 0)$ and $\int_S F \cdot dS = \int_0^{1/2} \int_0^1 (-V, 0, W) \cdot (1, 0, 0) \, dudv = -V/2$. You catch more rain if you drive faster.

PROPERTY ON THE MOON. Last year, in lack of a better present idea, I bought property on the moon (30 Us-Dollars for 1777.58 Acres (size of Manhattan)) at the Lunar Embassy (www.moonshop.com). An acre is now sold for 15 Dollars.

The coordinates of the parcel are $34E, 26N$. The moon has a radius of $r=1737$ km. The surface is $3.810^6 km^2$. Multiply this by 247 gives about 10 billion (10^{10}) acres (more than an acre for each person on the world). The moon surface is parameterized by $X(u, v) = (r \cos(u) \sin(v), r \sin(u) \sin(v), r \cos(v))$ where u is the longitude and $\pi/2 - v$ is the latitude. If we look at the moon from the direction $(1, 0, 0)$, how big does the area of the real estate $S = X([a, b] \times [c, d])$ appear?

Lunar Deed

From the recognized authority of the Lighted Lunar Surface, this document represents the issuing of 10 (ten) acres on the Lighted Lunar Surface.

This deed is for the Lunar Property listed below.

Lunar Description

Area E-10, Quadrant Echo, Lot Number 446
This property is located $34E, 26N$ squares south and $226E$ squares east of the extreme northwest corner of the recognized Lunar Grid.
(Ten acres equals approximately 1,777.58 acres)

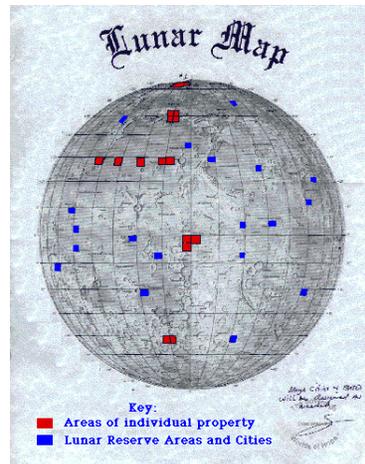
This deed is recorded in the Lunar Embassy record currently in the State, California, United States of America, and in the name of Dennis M. Knill from hereinafter known as owner of the above described property.

This deed is transferable or assignable upon the decision of the owner at his/her discretion.

This document conforms to all of the Lunar Real Estate Regulations set forth by the Head Cheese, (Dennis M. Knill) and shall be considered in proper order when his signature and seal are affixed to this document.

This sale approved by the Board of Realtors,
The Omnipotent Ruler of the Lighted Lunar Surface.

 12-18-2000
The Head Cheese Date



The flux of the vector field $(1, 0, 0)$ through the surface is $\int_S F \cdot dS = \int_a^b \int_c^d r^2 \cos(u) \sin^2(v) \, dv \, du$. This is also the area we see. You notice that the visible area is small for $\cos(u)$ small or $\sin(v)$ small, which is near $90E, 90W$ or near the poles.

CURL. In three dimensions, the curl of a vector field $F = (P, Q, R)$ is the vector field

$$\text{curl}(F)(x, y, z) = \nabla \times F = (R_y - Q_z, P_z - R_x, Q_x - P_y).$$

The curl is a vector measures the vorticity of the field. (Note that in two dimensions, the curl was a scalar! (In the plane, a rotation around a point is determined a single number, the angle). In three dimensions, the curl is a vector. (In space a rotation is described by a vector giving direction and encoding the angle with the length.)

EXAMPLE. The curl of the vector field $F(x, y, z) = (-y, x, z)$ is $(0, 0, 2)$.

FLUX OF CURL. Assume S is region in the plane and F is a vector field with zero z component: $F(x, y, z) = (P(x, y), Q(x, y), 0)$. The curl of F is the vector field $(0, 0, Q_x - P_y)$. The surface $X(u, v) = (u, v, 0)$ has $X_u \times X_v = (0, 0, 1)$ so that $F(X(u, v)) \cdot (X_u \times X_v)(u, v) = Q_x(u, v) - P_y(u, v)$. By Green's theorem, this is the line integral along the boundary.

Preview: the flux of the curl through a surface will be in general the line integral along the boundary of the surface.

DIVERGENCE. The divergence of a vector field in the plane is the scalar field $\text{div}(F)(x, y) = F_x + F_y$. The divergence of a vector field in space is $\nabla \cdot F(x, y, z) = \text{div}(F)(x, y, z) = F_x(x, y, z) + F_y(x, y, z) + F_z(x, y, z)$. The divergence is a measure of the "source" of the vector field.

Preview. We will see that the flux of a vector field F through a closed surface is the same as the total divergence of F inside the surface.

EXAMPLE. The divergence of the vector field $F(x, y, z) = (2x^2 + y, x, zx)$ is $5x$.

NABLA CALCULUS. Treat $\nabla = (\partial_x, \partial_y, \partial_z)$ as a vector. If f is a scalar and F is a vector field:

$$\begin{array}{ll} \nabla f &= \text{grad}(f) = f' \\ \nabla \times F &= \text{curl}(F) \\ \nabla \cdot F &= \text{div}(F) \end{array} \qquad \begin{array}{ll} \nabla \times \nabla f &= \text{curl grad } f = 0 \\ \nabla \cdot \nabla \times F &= \text{div curl } F = 0 \\ \nabla \cdot \nabla f &= |\nabla|^2 f = \Delta f = f_{xx} + f_{yy} + f_{zz} \\ \nabla \times (\nabla \times F) &= \text{curl curl } F = \text{grad div}(F) - \Delta F \end{array}$$

The first identity follows from $a \times a = 0$, the second from $(a \cdot a \times b) = 0$ (volume of parallelepiped spanned by a, a, b), the third is a consequence of $a \cdot a = |a|^2$, the fourth uses the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. We can prove such identities directly (and more painfully) by plugging in the definitions (see homework).

WHERE CAN FLUX INTEGRALS APPEAR?

- **Fluid dynamics** The flux of the velocity vector field of a membrane through a surface is the volume of the fluid which passes through that surface in unit time. Often one looks at incompressible fluids which means $\text{div}(v) = 0$. In this case, the amount of fluid which passes through a closed surface like a sphere is zero because what enters also has to leave the surface.
- **Electromagnetism** The change of the flux of the magnetic field through a surface is related to the voltage at the boundary of the surface. The magnetic field has zero divergence $\text{div}(B) = 0$ (there are no **magnetic monopoles**).
- **Magnetohydrodynamics** Magnetohydrodynamics deals with electrically charged "fluids" like plasmas in the sun or solar winds etc. The velocity field of the wind induces a current and the flux of the field through a surface can be used to calculate the magnetic field.
- **Gravity** We will see that the flux of a field through a closed surface like the sphere is related to the amount of source inside the surface. In gravity, where F is the gravitational field, the flux of the gravitational field through a closed surface is proportional to the mass inside the surface. Mass is a "source for gravitational field".
- **Thermodynamics.** The heat flux through a surface like the walls of a house are related to the amount of heat which is produced inside the house. A badly isolated house needs has a large heat flux and needs a large source of heating.
- **Particle physics.** When studying the collision of particles, one is interested in the relation between the incoming flux and out-coming fluxes depending on the direction. Key word: "differential cross-section".
- **Optics.** Examples are measurements of "Light fluxes". Important for lasers, astronomy, photography.