REMINDERS. The curl of a vector field F is

$$\text{curl}(P, Q, R) = \nabla \times F = (R_y - Q_z, P_z - R_x, Q_x - P_y) .$$

The flux integral of a vector field F through a surface $S = X(R)$ is defined as

$$\int\int_R F(X(u, v)) \cdot X_u \times X_v \, dudv$$

The line integral of a vector field F along a curve $\gamma = r([a, b])$ is given as

$$\int_a^b F(r(t)) \cdot r'(t) \, dt .$$

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library.

STOKES THEOREM. Let S be a surface with boundary γ and let F be a vector field. Then

$$\int\int_S \text{curl}(F) \cdot dS = \int_\gamma F \cdot ds .$$

Note: the orientation of γ is such that if you walk along the surface (head into the direction of the normal $X_u \times X_v$), then you have the surface is to the left.

EXAMPLE. Let $F(x, y, z) = (-y, x, 0)$ and let S be the upper unit hemisphere, then $\text{curl}(F)(x, y, z) = (0, 0, 2)$. The surface is parameterized by $X(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ on $R = [0, 2\pi] \times [0, \pi/2]$ and $X_u \times X_v = \sin(v)X(u, v)$ so that $\text{curl}(F)(x, y, z) \cdot X_u \times X_v = \sin(v)^2 2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} 2\sin(v)^2 \, dvdu = 2\pi$.

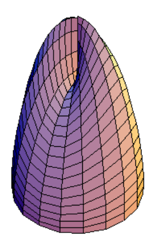
The boundary γ of S is parameterized by $r(t) = (\cos(t), \sin(t), 0)$ so that $ds = r'(t)dt = (-\sin(t), \cos(t), 0)dt$ and $F(r(t)) \, r'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_\gamma F \cdot ds$ along the boundary is 2π.

SPECIAL CASE: GREEN'S THEOREM. If S is a surface in the $x - y$ plane and $F = (P, Q, 0)$ has zero z component, then $\text{curl}(F) = (0, 0, Q_x - P_y)$ and $\text{curl}(F) \cdot dS = (Q_x - P_y) \, dxdy$.

CONSEQUENCE. The flux of the curl of a vector field through a surface depends only on the boundary.

EXAMPLE. Calculate the flux of the curl of $F(x, y, z) = (-y, x, 0)$ through the surface given by $X(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v) * \sin^2(u + \pi/2))$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

For every surface bounded by γ the flux of $\text{curl}(F)$ through through the surface is the same!



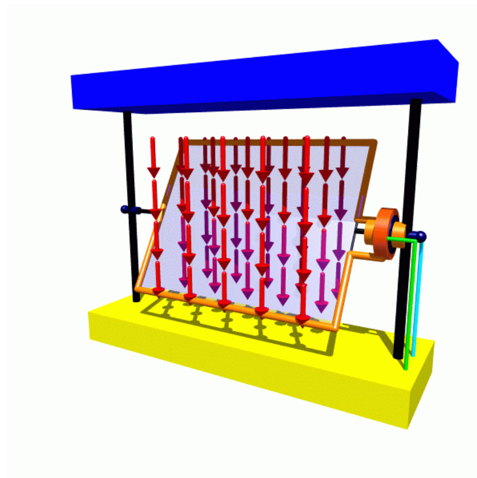
Compare this with:

For every curve between two points A, B the line integral of $\text{grad}(U)$ along gamma is the same!

PROOF OF STOKES THEOREM. For a surface which is a parallelepiped Stokes theorem is seen with Green's theorem: the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\text{curl}(F)$. For a general surface, we approximate the surface through a mesh of small parallelepipeds. When summing up line integrals along all these parallelepipeds, most cancel and only the integral along the boundary stays. The sum of the fluxes of the curl through boundary becomes the flux through the surface.

BIOT-SAVARD LAW. A magnetic field B in absence of an electric field satisfies a Maxwell equation $\mathrm{curl}(B) = (4\pi/c)j$, where j is the current and c is the speed of light. How do we get the magnetic field B, when the current is known? Stokes theorem can give the answer: take a closed path γ which bounds a surface S. The line integral of B along γ is the flux of $\mathrm{curl}(B)$ through the surface. By the Maxwell equation, this is proportional to the flux of j through that surface. Simple situation. Assume j is contained in a wire of thickness r which we align on the z-axes. To measure the magnetic field at distance $R > r$ from the wire, we take a curve $\gamma : r(t) = (R\cos(t), R\sin(t), 0)$ which bounds a disc S and measure $2\pi RB = \int_\gamma B \cdot ds = \int\int_S \mathrm{curl}(B)\ dS = \int\int_S 4\pi/cj\ dS = 4\pi J/c$, where J is the total current passing through the wire. The magnetic field satisfies $B = 2J/(cR)$. This is called the Biot-Savard law.

THE DYNAMO, FARADEY'S LAW. The electric field E and the magnetic field B are linked by a Maxwell equation $\mathrm{curl}(E) = -1/c\dot{B}$. Take a closed wire γ which bounds a surface S and consider $\int\int_S B \cdot dS$, the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \int\int_S B{\cdot}dS = \int\int_S \dot{B}{\cdot}dS = \int\int_S -c\mathrm{curl}(E){\cdot}dS = -c\int_\gamma E\ ds = U$, where U is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire, we get an electric voltage. That is what happens in a power-generator like an alternator in a car. (It is more efficient to move around the wire inside a fixed magnet).



THE MAXWELL EQUATIONS. B is the magnetic field, E the electric field, c the speed of light, write $\partial E/\partial t = E_t$, j is an electric current field and ρ is the charge (a scalar field). Remember $\mathrm{div}(F) = F_x + F_y + F_z$ is a scalar field.

$\mathrm{curl}(E) = -\frac{1}{c}B_t$	$\mathrm{div}(B) = 0$
$\mathrm{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	$\mathrm{div}(E) = 4\pi\rho$

They look only complicated because they are written in three dimensions. In four dimensions, where one writes current and charge together in a 4-vector $\mathbf{j} = (\rho, j)$ and the electric and magnetic field together in a four dimensional electromagnetic field tensor $F_{ik} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} = (E, B)$. The first group of equations can be written as $\partial_{x_l} F_{ij} + \partial_{x_i} F_{kl} + \partial_{x_k} F_{li}$. The second group of equations becomes $\sum_k \partial_{x_k} F^{ik} = -\frac{4\pi}{c} j^i$, where $F^{ik} = (-E, B)$ is as above.

Simplifying notation even more the Maxwell equations are written as $dF = 0, d^*F = j$. With magnetic monopoles, these equations would became even more symmetric $dF = i, d^*F = j$ but there is no evidence for magnetic monopoles.

STOKES THEOREM IN HIGHER DIMENSIONS. $\int\int_S \mathrm{curl}(F){\cdot}dS = \int_\gamma F{\cdot}ds$. Useful in special relativity (4D).

LINEINTEGRAL. Line integrals are defined in the same way in higher dimensions. $\int_\gamma F \cdot ds$, where $\cdot$ is the dot product in d dimensions and $ds = r'(t)dt$ is a small velocity element.

CURL AND ROTATION are related: the matrix $A_{ij} = \partial_j F_i - \partial_i F_j$ is skew symmetric $A_{ij} = -A_{ji}$ (compare: the symmetric Hessian $H_{ij} = H_{ji}$). The matrix $R = e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!}..$ is a rotation because $(u, v) = (e^A e^{-A} u, v) = (e^A u, e^{-A^T} v) = (e^A u, e^A v) = (Ru, Rv)$ shows that R preserves angle $\arccos(\frac{(u,v)}{|u||v|})$ and length $|u|^2 = (u, u)$.

CURL. In dimension d, the curl is a field $\mathrm{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components.

SURFACE INTEGRAL. In d dimensions, a surface element in the $i - j$ plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \mathrm{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \mathrm{curl}(F)_{ij} dS_{ij}$. If S is given by a map X from a planar domain R to $\mathbb{R}^d$, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u, v) = (U_i V_j - U_j V_i)\ dudv$.