SUGGESTED PROBLEMS.

- Section 5.7: pgs 300-301 numbers 1, 7, 11
- Use the divergence theorem to compute the flux of the following vector fields outward through the surface of the ball where $x^2 + y^2 + z^2 \leq 1$:
  - $F = (y^2 - 2x, x + z, \cos(y) + z)$
  - $F = (\sin(z - y^2), y^2 + 2y + z^4, x^4 - 1)$
- Suppose that $F$ is a vector field in space with $\text{div}(F) = 3$ and that is, at all points, where $z = 0$, tangent to the xy-plane. Compute the flux of $F$ outward through the $z \geq 0$ portion of the surface of the ball where $x^2 + y^2 + z^2 \leq 1$.
- Write down a vector field with vanishing divergence and with flux equal to $\pi$ through the $z \geq 0$ portion of the surface of the ball, where $x^2 + y^2 + z^2 \leq 1$.
- Write down a vector field whose curl is equal to $(1, 0, 0)$. Exhibit such a vector field whose path integral around the circle where $x = 0$ and $y^2 + z^2 = 1$ is equal to $2\pi$; or else explain why there aren't any.
- Exhibit a vector field,$F$, with the following property: Whenever $C$ is a circle on which $y$ is constant (so parallel to the xz-plane), then the path integral of F around the circumference of C is equal to the square of C's radius.
- Exhibit a vector field, F, with the following property: Whenever $C$ is a circle on which $y$ is constant (so parallel to the xz-plane), then the path integral of F around the circumference of $C$ is equals the constant value of y times the square of $C$'s radius.
- Write down a vector field whose components are not constant, but that has zero curl and zero divergence.
- Write down a vector field whose divergence is not everywhere zero, but whose flux through the surface of the ball where $x^2 + y^2 + z^2 \leq 1$ is zero.
- Write down a vector field whose path integral is zero around all circles with z = 0 and $x^2 + y^2 = constant$, but whose curl has component along $(0, 0, 1)$ which is not everywhere zero.
- Explain why Green's theorem is a special case of Stokes' theorem.

HOMEWORK. pages 300-301, numbers 4,6,8,10. Also do:

1) In each case, compute the line integral of the given vector field **F** about the boundary curve, $\gamma$, of the triangle in the plane $x + y + z = 1$ where x, y and z are all positive. In each case, traverse the $\gamma$ clockwise as viewed from $z = 100$ on the z -axis. Do this by first parameterizing each of the three segments of $\gamma$. Then, do the calculation via Stokes' theorem.
   a) $\mathbf{F} = (z, x, y)$.
   b) $\mathbf{F} = (x^2, y, z)$.
   c) $\mathbf{F} = (xy, z, 0)$.
   (In the physics literature, **F** would be the vector potential, **A**, for a magnetic field, **B** = curl(**A**). Thus, the application here of Stokes' theorem relates the flux of the magnetic field to the line integral of the vector potential.)
2) In each case, find the flux of the indicated vector field **F** through the boundary of the cube in $\mathbb{R}^3$ where $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$. Do the calculation first by computing the flux through each face, and then use the divergence theorem.
   a) $\mathbf{F} = (1, 1, 1)$.
   b) $\mathbf{F} = (x, 0, 0)$.
   (In the physics literature, **F** in this case would be the electric field, **E**, and then the flux integral computes the amount of electric flux through the given surface. The application of the divergence theorem in this case is known in physics as Gauss' law.)
3) Suppose that **F** is a vector field with curl $\mathbf{F} = (1, 1, 1)$. Find a plane in $\mathbb{R}^3$ with the property that the line integral of **F** about all closed curves in the plane is zero. (As in the first problem, **F** would be denoted by **A** in the physics literature, a vector potential for a magnetic field, **B** = curl(**A**).)
4) Let **F** be a vector field on $\mathbb{R}^3$ and let g be a function on $\mathbb{R}^3$. Prove the following:
   a) $\mathrm{div}(g\,\mathbf{F}) = g\,\mathrm{div}(\mathbf{F}) + \nabla g \bullet \mathbf{F}$.
   b) $\mathrm{curl}(g\,\mathbf{F}) = g\,\mathrm{curl}(\mathbf{F}) + \nabla g \times \mathbf{F}$.
5) Suppose that **F** is a vector field on $\mathbb{R}^3$ and $|\mathbf{F}| \le 1$ at all point inside the ball V where $x^2 + y^2 + z^2 \le 1$. Explain why $-4\pi \le \iiint_V \mathrm{div}(\mathbf{F}) \le 4\pi$. (Hint: Use the divergence theorem to rewrite this integral and then think about the size of the resulting integrand.) (Once the physics literature would use **E** instead of **F** when talking about an electric field, and then this would be another question concerning Gauss law.)
6) Take **F** as in the previous problem and let R denote the disc $x^2 + y^2 \le 1$ in the $z = 0$ plane and let $\mathbf{n} = (0, 0, 1)$ be its normal. Explain why $-2\pi \le \iint_R \mathrm{curl}(\mathbf{F})\bullet\mathbf{n}\, dS \le 2\pi$. (In physics, this problem would use **A** for **F** and call **A** a vector potential. Then, curl(**A**) is the resulting magnetic field. In this language, the problem asks about the strength of the magnetic field flux.)
7) Compute the volume of the tetrahedron V (four sided prism) in $\mathbb{R}^3$ given by $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x + y + z \le 1$. (Thus, the boundary of V has four triangular faces, one each in the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.) Do the computation first as an iterated integral, and then via the divergence theorem as a flux integral through the boundary of V of the vector field $\mathbf{F} = (x, 0, 0)$.
8) For each angle $\theta$ between 0 and $2\pi$, let $D_\theta$ denote the disk in $\mathbb{R}^3$ whose radius is 1, center is the origin in $\mathbb{R}^3$ and which lies in the plane $\cos(\theta)\, x + \sin(\theta)\, z = 0$. Let $I(\theta)$ denote the absolute value of the line integral over the boundary of $D_\theta$ of the vector field $\mathbf{F} = (0, 0, y)$. What is the maximum value of $I(\theta)$ and what are the angles $\theta$ which have this value? (Hint: The problem is easier if you use Stokes' theorem and think about how the size of $\mathrm{curl}(\mathbf{F})\bullet\mathbf{n}$ depends on the angle $\theta$.)