

TWO THEOREMS. "I would like to share with you a revelation..." -Agent Smith in the Matrix.

EXISTENCE OF A POTENTIAL. If G is a region in the plane which has no holes and F is a vector field satisfying $\text{curl}(F) = 0$, then there exists a function U called **potential** such that $F = \nabla U$. A similar thing is true in space: if G is a region in space in which every closed curve can be contracted to a point, then $\text{curl}(F) = 0$ implies that there exists a potential U such that $F = \nabla U$.

PROOF. Fix a point y in G and define $U(x) = \int_{\gamma(x)} F(x) \cdot ds$. By Green's theorem, the value of U is independent of the path $\gamma(x)$ connecting y with x . In three dimensions, use Stokes theorem for a surface obtained when γ is deformed to a point. By taking directional derivatives of U into the main directions, we see that $F = \nabla U$.

EXISTENCE OF A VECTOR POTENTIAL. Let $F = (P, Q, R)$ be a vector field satisfying $\text{div}(F) = 0$ everywhere, then there exists a vector field A called **vector potential** such that $F = \text{curl}(A)$.

PROOF. Take $A = (\int_0^z Q(x, y, t) dt - \int_0^y R(x, t, 0) dt, -\int_0^z P(x, y, t) dt, 0)$.

APPLICATIONS. If E is the electric field which does not change in time, then it satisfies $\text{curl}(E) = 0$. There exists then a potential ϕ such that $E = \text{grad}(\phi)$.

If B is the magnetic field, then it satisfies $\text{div}(B) = 0$. There exists therefore a vector potential A such that $B = \text{curl}(A)$.

NABLA CALCULUS. Working with $\nabla = (\partial_x, \partial_y, \partial_z)$ can simplify some computations. Just write $\nabla f = \text{grad}(f)$, $\nabla \times F = \text{curl}(F)$, $\nabla \cdot F = \text{div}(F)$, $|\nabla|^2 f = \Delta f$ and use ∇ as a vector and Leibniz rule like in $\nabla(fg) = \nabla f g + f \nabla g$. Examples: E, F are vector fields, f, g are functions:

- $\text{div}(\text{curl})(F) = 0$ follows from $\nabla \cdot (\nabla \times F) = 0$. The following identities are not so obvious:
- $\text{curl}(\text{grad})(f) = 0$ follows from $\nabla \times \nabla f = 0$.
- Green's first identity $\int \int \int_G (f \Delta g + \nabla f \cdot \nabla g) dV = \int \int_S f \nabla g \cdot dS$ follows from $\nabla \cdot (f \nabla g) = (f \Delta g + \nabla f \cdot \nabla g)$ and Gauss theorem.
- Green's second identity, $\int \int \int_G (f \Delta g - g \Delta f) dV = \int \int_S (f \nabla g - g \nabla f) \cdot dS$ follows from $\nabla \cdot (f \nabla g - g \nabla f) = (f \Delta g - g \Delta f)$ and Gauss theorem.
- $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \Delta F$.
- $\nabla(E \cdot F) = (E \cdot \nabla)F + (F \cdot \nabla)E + E \times (\nabla \times F) + F \times (\nabla \times E)$
- $\nabla \cdot (E \times F) = F \cdot (\nabla \times E) - E \cdot (\nabla \times F)$
- $\nabla \times (E \times F) = E(\nabla \cdot F) - F(\nabla \cdot E) + (F \cdot \nabla)E - (E \cdot \nabla)F$.

Note: Except maybe $\text{div}(\text{curl})(F) = 0$ and $\text{curl}(\text{grad})(F) = 0$, these are not things to learn by heart but to have somewhere in a collection of formulas. Books like "Jackson's book "Classical Electrodynamics", the "bible" in electrodynamics contains such formulas on the first or last page, where they can be looked up easily.

INTERPRETATION OF PDE's. The integral theorems relate partial differential equations (microscopic laws) with physical interpretations (macroscopic laws).

EXAMPLE: ELECTRODYNAMICS. "Sentinels! Tank, charge the EMP!" Morpheus, in the movie "the Matrix"

Maxwell equation	$\text{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
Maxwell equation	$\text{curl}(E) = -\frac{1}{c} B_t$	Faraday's law	change of magnetic flux induces voltage
Maxwell equation	$\text{curl}(B) = \frac{1}{c} E_t + \frac{4\pi}{c} j$	Ampère's law	current or change of E produces magnetic field
Maxwell equation	$\text{div}(E) = 4\pi \rho$	Gauss law	electric charges are sources for electric field

EXAMPLE: FLUID DYNAMICS. v velocity, ρ density of fluid. "Do you think that is air, you are breathing, Neo?" -Morpheus.

Continuity equation	$\dot{\rho} + \text{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\text{div}(v) = 0$	incompressible fluids, in 2D: $v = \text{grad}(u)$
Irrotational	$\text{curl}(v) = 0$	irrotation fluids

EXAMPLE: THERMODYNAMICS. j : heat current, T temperature. "Welcome to the real world!", –Morpheus.

Heat equation	$T = \kappa \Delta T$	$j = \kappa \text{grad}(T)$ so that $\text{div}(j) = -\kappa \Delta T$. Apply the continuity equation.
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INCARNATIONS OF ONE THEOREM. All the integral theorems are of the form $\int_G dF = \int_{\delta G} F$, where dF is the "derivative" of F and δG is the "boundary" of G . In n dimensions, there are n theorems. They are labeled by the dimension of G .

THE DERIVATIVE d . grad, curl or div look very different but are closely related.

1D	$f \mapsto f'$			
2D	$f \mapsto \nabla f$	$F \mapsto \text{curl}(F)$		
3D	$f \mapsto \nabla f$	$F \mapsto \text{curl}(F)$	$F \mapsto \text{div}(F)$	
4D	$f \mapsto \nabla f$	$F \mapsto d(F)$	$F \mapsto d^*(F)$	$F \mapsto \text{div}(F)$

Don't **think** you are, **know** you are." – Morpheus in the movie: "the matrix".

One can actually define d in any dimensions. There are n such derivatives in dimension n .

THE BOUNDARY δ . "This is the sound of inevitability". – Agend Smith in the movie "the matrix".

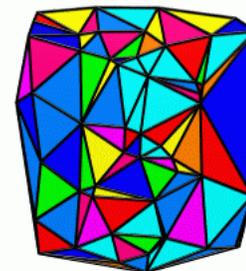
1D	interval \mapsto endpoints			
2D	curve \mapsto endpoints	region \mapsto curve		
3D	curve \mapsto endpoints	surface \mapsto curve	solid \mapsto boundary surface	
4D	curve \mapsto endpoints	surface \mapsto curve	3D solid \mapsto boundary surface	4D space \mapsto 3D boundary

OVERVIEW INTEGRAL THEOREMS. Here is the **periodic system of integral theorems**. The 4D version belongs already to the "heavy elements": these theorems do not appear very often.

1D	fundamental theorem of calculus			
2D	fundamental theorem of line integrals	Green's theorem		
3D	fundamental theorem of line integrals	Stokes theorem	Gauss theorem	
4D	fundamental theorem of line integrals	Stokes theorem	Dual Stokes theorem	Gauss theorem

RELATIONS WITH DISCRETE MATH. The multi-variable results can be adapted other situations. For example, consider a network modeled by a planar graph in which all edges bound triangles. A **scalar function** assigns a value f_n to each node n . A **area function** assigns values f_T to each triangle T . A **vector field** assign values F_{nm} to each edge connecting node n with node m . The **gradient** of a scalar function is the vector field $F_{nm} = f_n - f_m$. The **curl** of a vector field F is attaches to each triangle (k, m, n) the value $\text{curl}(F)_{kmn} = F_{km} + F_{mn} + F_{nk}$. It is a measure for the circulation of the field around a triangle. A curve γ in our discrete world is a set of points $r_j, j = 1, \dots, n$ such that nodes r_j and r_{j+1} are adjacent. For a vector field F and a curve γ , the **line integral** is $\sum_{j=1}^n F_{r(j)r_{j+1}}$. A **region** R in the plane is a collection of triangles T . The **double integral** of an area function f_T is $\sum_{T \in R} f_T$. The **boundary** of a region is the set of edges which are only shared by one triangle. The orientation of γ is as usual.

"Free your mind, Neo!"
(Morpheus in "the matrix")



DISCRETE FUNDAMENTAL THEOREM OF LINE INTEGRALS. If $F = \text{grad}(U)$ and $\gamma = \{r_j\}_{j=1}^n$ is a curve in the graph, then $\sum_{j=1}^n F_{r_j r_{j+1}} = U_{r_n} - U_{r_0}$.

DISCRETE GREEN THEOREM. Given a discrete region R with boundary $\gamma = \{r_j\}_{j=1}^n$ and a vector field F , then $\sum_{j=1}^n F_{r_j r_{j+1}} = \sum_{T \in R} \text{curl}(F)_T$.

IDENTITIES. You can check that $\text{curl}(\text{grad}(F)) = 0$ as in the continuum case.

It is fun to formulate the theorems of Stokes and Gauss for a discrete graph in three dimensional space. You will see that there are different type of vector fields: vector fields which are attached to edges and vector fields which are attached to polygons forming faces of three dimensional polytopes. The gradient of a scalar function is a vector field attached to edges, the curl of a vector field is a vector field attached to faces. The divergence of a vector field attached to faces is a scalar field attached to three dimensional polytopes.