Differential equations are laws which determine how a system evolves in time. An example of a system is our solar system. The motion of the planets is described by Newton equations, an ordinary differential equation (ODE). An other example is the electromagnetic field for a fixed charge and current distribution. The motion of this system is described by Maxwell equations. In vacuum, they reduce to the wave equation, describing electromagnetic waves like light. The Maxwell equations or the wave equations are examples of partial differential equations (PDE's).

ODE. An **ordinary differential equation** (ODE) is an equation relating derivatives $f'(t), f''(t), ...$ of a function f or a curve $t \mapsto f(t)$. Only derivative with respect to one variable can appear. Otherwise, the equation is called a **partial differential equation** (PDE). If derivatives up to the order n appear, the equation is called a **differential equation of order** n. If $f(t)$ is a vector in $\mathbf{R}^d$, then the ODE is in d dimensions:

Equation	ODE?
$f''(t) + \cos(f(t)\sin(t)) = 0$	yes
$f_x(x,y) + f_y(x,y) = 0$	no
$\ddot{x}(t) = \sin(x(t))$	yes
$(\dot{x}, \dot{y}) = (-y, x)$	yes
$\mathrm{curl}(F) = A$	no

Equation	order	dim
$f''(x) = f(x)$	2	1
$\sin(f'''(x)) = f'(x)$	3	1
$(f'(x), g'(x), h'(x)) = F(f(x), g(x), h(x))$	1	3
$(\ddot{x}, \ddot{y}) = (-y, x)$	2	2
$(\dot{x})^5 = x$	1	1

While for usual equations like $x^2 + 2x + 1 = 0$, the unknown x is a **number**, for a differential equation like $\ddot{x} = x$ it is a **function** $x(t)$ respectively a **curve** $t \mapsto \vec{x}(t)$. Think always of t as **time**.

SOLVING AN ODE. Solving an ODE means to find all functions or curves which satisfy the equation. Examples.

$f'(x) = f(x)$ has solutions e^x or $5e^x$ and $f''(x) = -f(x)$ has solutions $\cos(x)$ or $4\sin(x)$ and $(x'(t), y'(t)) = (2, x(t))$ has solutions $(2t, t^2)$.

FACT. Every ODE can be written as $d/dt\vec{x}(t) = \vec{F}(\vec{x}(t))$, where $\vec{F}$ is a vector field and $\vec{x}$ is a vector:
PROOF. Solve $F(f, f', f'', \dots, f^{(n)}) = 0$ for $f^{(n)} = G(f, f', f'', \dots, f^{(n-1)})$ and introduce the vector $\vec{x} = (x_1, x_2, \dots, x_n) = (f, f', f'', \dots, f^{(n-1)})$. Then $\vec{x}' = (x_2, x_3, \dots, G(x_1, \dots, x_{n-1})) = F(\vec{x})$.

Example: $f'''(t) - \sin(f''(t)) + f(t) = 0$. Define $\vec{x} = (x(t), y(t), z(t)) = (f''(t), f'(t), f(t))$. Then $d/dt\vec{x} = (f'''(t), f''(t), f'(t)) = (\sin(f''(t)) - f(t), f''(t), f'(t)) = (\sin(y) - x, y, x) = F(\vec{x})$. By introducing t as a separate variable, one can have F time-independent. F is then called autonomous. **By increasing the dimension, an ODE can be written as a first order autonomous system** $\dot{x} = F(x)$.

EXAMPLES OF ODE's.

$\dot{x}(t) = ax(t)$.	Population models for $a > 0$, radioactive decay for $a < 0$
$\dot{x} = ax(1-x)$.	Logistic equation: population model
$\ddot{x} = -ax$.	Harmonic oscillator: describes periodic oscillations
$m_j\ddot{x}_j = \sum_{i \neq j} m_i/\vert x_i - x_j\vert^2$	Newton equations. Example: x_j positions of planets of mass m_j.

SOLVING DIFFERENTIAL EQUATIONS. The different approaches are illustrated with $\dot{x} = x$.

1) **Integration.** To find explicit solutions of $dx/dt = x$ we write $dx/x = dt$ or $\int \frac{1}{x}dx = t + c$. Integration on both sides gives $\log(x) = t + c$ or $x(t) = e^t e^c = Ce^t$.
2) **Expansion**. For example by Taylor expansion: if $x(t) = a_0 + a_1 t + a_2 t^2 + ...$, then $\dot{x}(t) = a_1 + 2a_2 t + 3a_3 t^2 + ...$ Comparing coefficients gives $a_1 = a_0, a_2 = a_1/2 = a_0/2, a_3 = a_2/3 = a_0/6$ etc. so that $x(t) = a_0(1 + t/2 + t^2/3! + ...)$.
3) **Approximation**. Simplest numerical method is the linear approximation $x(t + dt) = x(t) + dt\dot{x}(t) = x(t) + dtx(t) = (1 + dt)x(t)$ and $x(t + kdt) = (1 + dt)^k x(t)$. If $dt = t/n$ and $k = n$, then $x(0 + t) = x(0 + nt/n) = (1 + t/n)^n x(0)$.

In the example, all three methods led to the correct solution $x(t) = Ce^t$. In 1), C was an integration constant, in 2), the zeroth Taylor coefficient a_0, in 3), $C = x(0)$. The explicit expression e^t came in 1) from anti-derivation, in 2) from $(1 + t/2 + t^2/3! + ...) = e^t$, in 3) from $\lim_{n\to\infty}(1 + t/n)^n = e^t$. For most ODE's, there are no closed form solutions. One can work however with truncated Taylor series or approximating polygons $x(dt), x(2dt), \dots$ for the path $x(t)$.

EXISTENCE OF SOLUTIONS of $\dot{x} = F(x)$. Given $x(0)$, there exists a solution $x(t)$ for some time $t \in [0, a]$ if F is smooth. Solutions don't always exist for all times: $\dot{x} = x^2$ can be solved by integrating $dx/x^2 = dt$ so that $-1/x = t + c$ or $x(t) = -1/(t+c)$. From $x(0) = -1/c$ we get $c = -1/x(0)$ and $x(t) = -1/(t - 1/x(0))$. The solutions exists for $t \leq 1/x(0)$.

THE EQUATION $\dot{x} = ax$. For $a > 0$, it models a explosive chain-reaction or population growth (the number of newborn people is proportional to the number of people which exist already). For $a < 0$, $\dot{x} = ax$ has solutions which decay (example is radioactive decay: as more atoms there are, as more atoms decay). The explicit solution is $x(t) = x(0)e^{at}$. For $a = 0$, there are only solutions x which are constant in time.

LINEAR EQUATIONS. A ODE is **linear** if it can be written as $\dot{x} = Ax$, where A is a matrix. It can be solved by $x(t) = e^{At}x(0)$, where the matrix e^B is defined as $e^B = 1 + B + B^2/2! + B^3/3! \dots$ and for example $B^3 = BBB$ is obtained by matrix multiplication. To understand the solutions, it is helpful to choose coordinates in which A is simple, for example diagonal. (See Math21b).

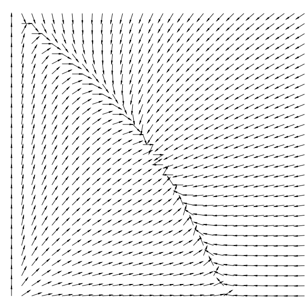
EXAMPLE. The ODE $\ddot{x} = -x$ is with $y = \dot{x}$ equivalant to $(\dot{x}, \dot{y}) = (y, -x)$ and can be written as $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$. Because $A^2 = -1, A^3 = -A, A^4 = 1, A^5 = A$ etc. we have $e^{At} = \begin{bmatrix} 1 - t^2/2! + t^4/3! - \dots & t - t^3/3! + t^5/5! - \dots \\ -t + t^3/3! - t^5/5! + \dots & 1 - t^2/2! + t^4/3! - \dots \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$ leading to an explicit solution $x(t) = \cos(t)x(0) + \sin(t)y(0), y(t) = -\sin(t)x(0) + \cos(t)y(0)$. Indeed, the circular curves $r(t) = (x(t), y(t))$ solve the differential equation.

SEPARATION OF VARIABLES. A system $\dot{x} = g(x,t)$ (equivalently $\dot{x} = g(x,t), \dot{t} = 1$) can sometimes be solved by **separation of variables**: $dx = f(x)g(t)dt$ or $dx/f(x) = g(t)dt$ can be integrated:

- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t)\, dt$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x)\, dx = \int_0^t g(t)\, dt = G(t)$ so that $x(t) = H^{-1}(G(t))$.

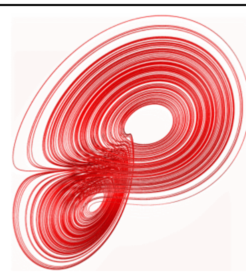
In general, there are no closed form solutions (i.e. $\dot{x} = e^{-t^2}$ has a solution $x(t) = \int_0^t e^{-t^2}\, dt$ which can not be expressed by $\exp, \sin, \log, \sqrt{\cdot}, etc.$. The anti-derivative of $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ is $x(t) = t - t^3/3 + t^4/(3 \cdot 2!) - t^7/(7 \cdot 3!) + \dots)$.

2D SYSTEMS. A two dimensional ODE is of the form $\dot{x} = P(x,y), \dot{y} = Q(x,y)$. If $r(t) = (x(t), y(t))$ is the solution curve, then the ODE tells that the velocity vector $r'(t)$ is equal to $F(r(t))$. Often a good understanding is achieved by drawing vector field and some solution curves.
Examples 2) and 3) below are 2D systems. Example 1) shows a system in three dimensions and also the timedependent system 4) became a 3D system when adding t as the z variable.



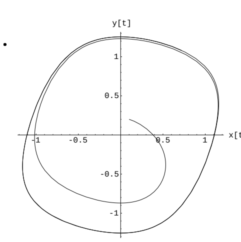
1) LORENTZ SYSTEM.

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3} \end{aligned}$$



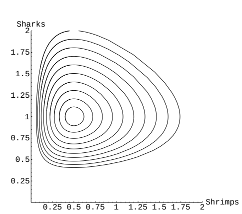
2) VAN DER POL EQUATION.
$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ or

$$\begin{aligned} \dot{x} &= y - (\frac{x^3}{3} - x) \\ \dot{y} &= -x \end{aligned}$$



3) VOLTERRA-LODKA SYSTEMS.

$$\begin{aligned} \dot{x} &= -\frac{y}{10} + \frac{2xy}{5} \\ \dot{y} &= \frac{2x}{5} - \frac{2xy}{5} \end{aligned}$$



4) THE DUFFING SYSTEM.
$\ddot{x} + \dot{x}/10 - x + x^3 - 12\cos(t) = 0$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{y}{10} - x + x^3 - 12\cos(z) \\ \dot{z} &= 1 \end{aligned}$$