PDE. If an equation for a function contains derivatives with respect to different variables, it is called PDE. We will see examples of PDES, see how they are derived from physics and how one can solve them.

EXAMPLES of partial differential equations:

- $u_t = u_x$ **advection equation**, models transport
- $u_{tt} = \Delta u = u_{xx} + u_{yy}$ **wave equation** in 2D.
- $u_t = \kappa u_{xx}$ **heat equation** in 1D
- $u_t = \kappa \Delta u = \kappa(u_{xx} + u_{yy})$ **heat equation** in 2D.
- $u_{tt} = \Delta u - m^2 u$ **Klein-Gordon** equation, appears in quantum theory.
- $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$ **Laplace equation** in 3D.
- $\text{grad}(U) = F$ search of **potential** U for vector field F. (Need $\text{curl}(F) = 0$).
- $\text{div}(B) = 0$, $\text{div}(E) = 0$, $\dot{B} = -c\ \text{curl}(E)$, $\dot{E} = c\ \text{curl}(B)$. **Maxwell equations** in vacuum.
- $\text{curl}(A) = F$ search of **vector potential** for vector field F. (Need $\text{div}(F) = 0$).
- $\Delta u = 4\pi\rho$ **Poisson equation**. ρ charge, $E = \text{grad}(u)$ is electric field.
- $u_t + u_{xxx} + 6uu_x = 0$ **KdV equation** has soliton solutions famous nonlinear waves.
- $i\hbar\ \dot{u} = \hbar^2 \Delta u/2m + Vu$ **Schrödinger equation** in quantum mechanics.
- $u_{tt} - u_{xx} + \sin(u) = 0$ **Sine-Gordon equation** has "kink-antikink" solutions.
- $\dot{u} + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \text{div}(u) = 0$ **Navier Stokes** equations. Win 1 Mio prove existence!
- $\dot{u} = uu_x$ **Burger equation**, models shocks, breaking waves.
- $\dot{u} = (u_x)^2/2 + V(x)$ Hamilton-Jacobi equation. Classical mechanics.
- $\det(f'') = \rho$ **Monge-Ampere equation**, f'' Hessian of $f = f(x, y, z)$. Appears in geometry.

DERIVATION OF HEAT EQUATION. Conservation of mass tells that the flux of a fluid with density ρ and speed j through a surface S bounding a region G is equal to the change of mass in the region $\int\int_S j \cdot dS = \int\int\int_G \rho\ dV$. By Gauss theorem, $\int\int_S j \cdot dS = \int\int\int_G \text{div}(j)\ dV$ so that $\int\int\int_G (\text{div}(j) - \rho)\ dV = 0$. Because the spherical region is arbitrary, we have $f = \text{div}(j) - \rho = 0$ (if the function f were not zero, we could chose a region G contained in $f > 0$). The density and the current are therefore linked by the **continuity equation** $\dot{\rho} = \text{div}(j)$. If particles move randomly, they tend to move more likely from places with high density ρ to places with low density. This means $j = \kappa\ \text{grad}(\rho)$. Plugging this into the continuity equation gives $\dot{\rho} = \kappa \text{div}(\text{grad})(\rho) = \kappa \Delta \rho$. This is the **diffusion equation**, also called **heat equation**.

DERIVATION OF THE WAVE EQUATION. The identity $\Delta B = \text{grad}(\text{div}(B)) - \text{curl}(\text{curl}(B))$ gives together with $\text{div}(B) = 0$ and $\text{curl}(B) = \dot{E}/c$ the relation $\Delta B = -d/dt \text{curl}(E)/c$ which leads with $\dot{B} = -c\ \text{curl}(E)$ to the wave equation $\Delta B = d^2/dt^2 B/c^2$. This equation $B_{tt} = c^2\ \Delta B$ (as well the equation $E_{tt} = c^2\ \Delta B$ derived in the same way) describes the propagation of electric and magnetic fields in vacuum.

DERIVATION OF THE TRANSPORT EQUATION. Start again with the continuity equation $\dot{u} = \text{div}(j) = j_x$ in one dimensions. If the current j is proportional to u_x we get $u_t = au_x$.

DERIVATION OF BURGER'S EQUATION. If $j = (u^2)_x$, this means that the current j depends nonlinearly on u, becoming large for large u. The corresponding equation $u_t = 2uu_x$ is a model for waves near the beach.

SUPERPOSITION. A PDE is called **linear** if it is linear in each variable, as well as in u, u_x, u_t etc. The heat, Laplace, wave, advection or Maxwell equations in vacuum are linear. The Burger equation, Sin-Gordon or KdE equation are examples of nonlinear equations.

If u and v are solutions of a linear equation, then λu and the superposition $u + v$ are solutions too.

THE ADVECTION EQUATION. A general solution of $u_t = cu_x - ru$ is

$$u(t, x) = e^{-rt} f(x + ct) ,$$

here $f(x) = u(0, x)$. This formula provides an explicit solution of a PDE. The initial function $u(0, x)$ determines $u(t, x)$ at later time in an explicit way.

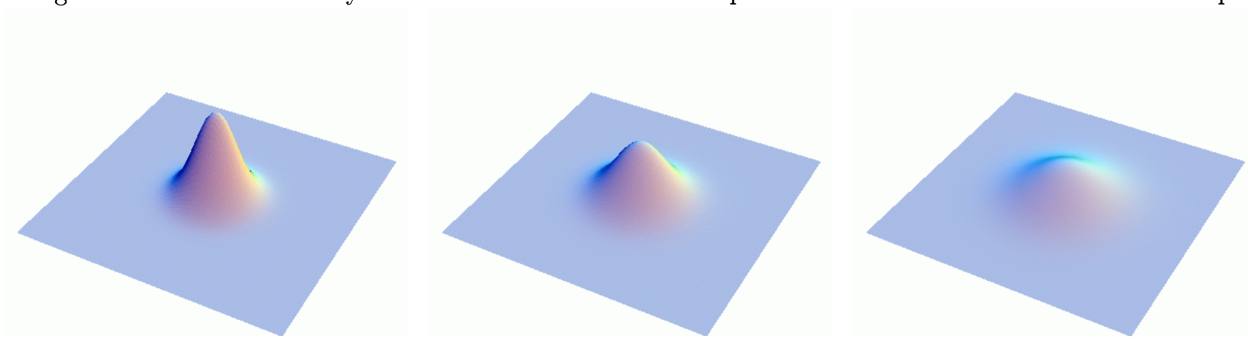
DIFFUSION EQUATION. A special solution to the diffusion equation $u_t = \mu u_{xx} - ru$ is

$$u(x,t) = a\frac{1}{\sqrt{4\pi\mu t}}e^{-rt}e^{-x^2/(4\mu t)} .$$

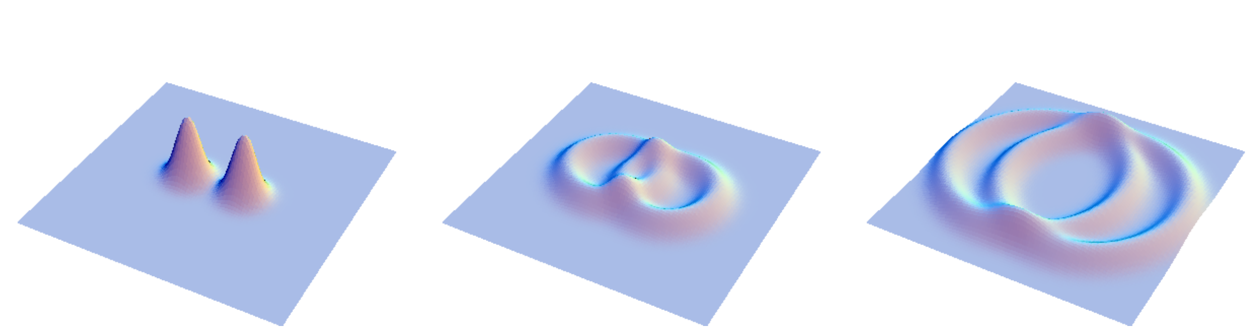
(Check $u_t/u = -r + (x^2 - 2\mu t)/(4\mu t^2), u_{xx}/u = (x^2 - 2\mu t)/(4\mu^2 t^2)$). Also translations $u(x-s,t)$ are solutions. By superposition, sums of such functions are solutions and by passing to limits of Riemann sums, also integrals

$$u(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty} a(s)e^{-rt}e^{-(x-s)^2/(4\mu t)}\, ds$$

is a solution. By taking the limit $t \to 0$, we see that $a(s) = u(0,s)$. We have therefore an explicit formula for the solution of the diffusion equation. The initial heat distribution determines the distribution at time t. It is considered an explicit formula. It can be generalized to arbitrary dimensions. The three pictures show an evolution in the plane.



WAVE EQUATION. Write $u_{tt} - u_{xx} = 0$ as $(\partial_t - \partial_x)(\partial_t + \partial_x)u$ or $(\partial_t + \partial_x)(\partial_t - \partial_x)u$. You see that if u satisfies the transport equation $(\partial_t + \partial_x)u = 0$ or $(\partial_t - \partial_x)u = 0$, then u solves the wave equation. Solutions are $f(x-t)$ or $g(x+t)$. Now, if $u(0,x) = F(x)$ and $u_x(u,x) = G(x)$ are known and $u(t,x) = f(x-t) + g(x+t)$, then $F(x) = u(0,x) = f(x) + g(x)$ and $G(x) = u_t(0,x) = -f'(x) + g'(x)$ determine f and g: from $F = f + g, F' = f' + g', G = g' - f'$ we get $f' = (F' - G)/2$ and $g' = (F' + G)/2$ and so f, g. The pictures show an evolution of the wave equation in two dimensions.



SOLUTIONS BY SEPARATIONS OF VARIABLE. Some linear equations in one dimension can be solved by starting with an "Ansatz" $u(t,x) = T(t)X(x)$.
TRANSPORT EQUATION. $u_t = \dot{T}X$, $u_x = TX'$. Because $\dot{T}/T = \dot{X}/X$ does not depend on x (look at LHS) and not depend on t (look at RHS), it is a constant C and we have $T(t) = e^{Ct}$ and $X(x) = e^{Cx}$ and therefore $u(t,x) = T(t)X(x) = e^{Ct}e^{Cx} = e^{C(t+x)}$. By summing up such solutions one build any solution $f(t+x)$.
DIFFUSION EQUATION. $u_t = \ddot{T}X$, $u_{xx} = TX''$. Again, $\dot{T}/T = X''/X$ is constant $C = -k^2$ so that $T(t) = e^{-k^2 t}$ and $X(x) = e^{ikx}$ and $u(t,x) = e^{-k^2 t}e^{ikx}$. (If X is a bounded function, then C can not be positive so that we can write $C = -k^2$.) A general solution is a superposition $u(t,x) = \int a(k)e^{-k^2 t}e^{ikx}dx$ of such solutions. One can determine $a(k)$ from the initial condition $u(0,x) = \int a(k)e^{ikx}dx$ using Fourier theory.
WAVE EQUATION. Also the wave equation can be treated like that. We get $\ddot{T}/T = X''/X = C$. Again $C = -k^2$ and $X(x) = e^{\pm ikx}, T(t) = e^{\pm ikt}$ so that $u(t,x) = e^{\pm ikx}e^{\pm ikt}$ is a solution. We see that $a(k)e^{ik(x-t)}$ and $b(k) = e^{ik(x+t)}$ are solutions. Superposition leads to arbitrary solutions $u(t,x) = f(x-t) + g(x+t)$.

BEYOND PDE's. A computer which solves ODE's or PDE's numerically discretizes the system. u_t is replaced by the difference $u_{n+1} - u_n$, u_{tt} by $u_{n+1} - 2u_n - u_{n-1}$. The equation $\dot{x} = ax$ for example is discretizes as $x_{n+1} - x_n = ax_n$ or $x_{n+1} = (1+a)x_n$. Instead of a **curve** $x(t)$, the solution is a **sequence** x_n. Time is discrete. Also PDE's can be discretized. The diffusion equation becomes $u_{n+1,m} - u_{n,m} = u_{n,m+1} - 2u_{n,m} + u_{n,m-1}$ which is $u_{n+1,m} = u_{n,m+1} - u_{n,m} + u_{n,m-1}$. This is also called a **coupled map lattice**. In each time step, a lattice configuration $\{u_{n,m}\}_{m=-\infty}^{\infty}$ is mapped into a new configuration. If also the values of $u_{n,m}$ are discretizes, one gets a **cellular automaton**. Both can be used to simulate partial differential equations numerically.