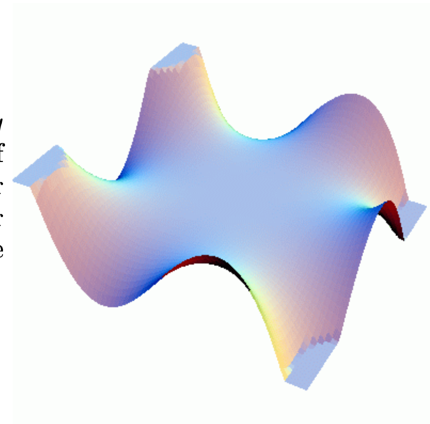
12/7/2001, PARTIAL DIFFERENTIAL EQUATIONS II — Math 21a, O. Knill

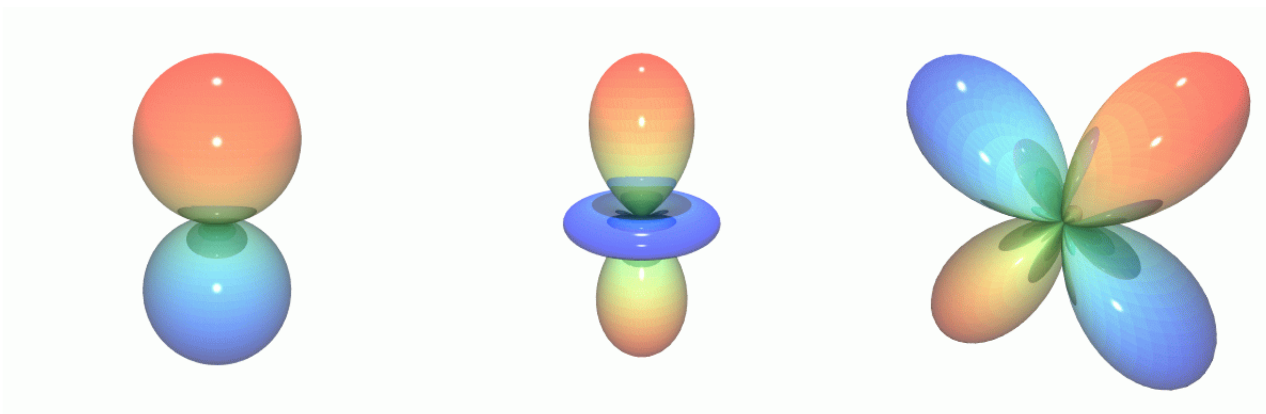
A great deal of my work is just playing with equations and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just looking for beautiful mathematical relations which maybe don't have any physical meaning at all. Sometimes they do.
Paul A. M. Dirac

LAPLACE EQUATION. In the plane, the equation is $\Delta u = \mathrm{div}\,\mathrm{grad} u = u_{xx} + u_{yy} = 0$. Solutions are called **harmonic**. You have found such solutions already in an earlier homework assignment.

Here is a general way to produce solutions: if $z = x + iy$ is a complex number, then the real or imaginary part of $(x+iy)^n = r^n e^{in\theta}$ are solutions of the Laplace equation. For example, $u(x,y) = \mathrm{Re}((x+iy)^3) = x^3 - 3xy^2 = r^3\cos(3\theta)$ or $u(x,y) = \mathrm{Im}((x+iy)^3) = 3x^2y - y^3 = r^3\sin(3\theta)$ solve the Laplace equation.



In one dimension, only linear functions $u(x) = ax + b$ are harmonic: just solve $u_{xx} = 0$. In three dimensions, harmonic functions are constructed in spherical coordinates $u(r,\phi,\theta) = \mathrm{Re}(r^l P_l^m(\cos(\theta))e^{im\phi})$ with special polynomials P_l. The angular portions of such functions are called **spherical harmonics** and denoted by Y_{lm}. The pictures show three examples. These functions are important for quantum mechanics because there are functions R_{nl} of r such that $u_{n,l,m}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\phi,\theta)$ define stationary electron distributions in the hydrogen atom. Knowing about these functions allows to understand of the hydrogen atom and makes the construction of the periodic system of elements transparent.



You can play with spherical harmonics in a Java applet on *http://www.sct.gu.edu.au/research/laserP/livejava/spher_harm.html* Harmonic functions have the property that they are stationary solutions to the heat equation $u_t = \mu\Delta u$ or the wave equation $u_{tt} = c^2\Delta u$ or the Schrödinger equation of a free particle.

H																	He
Li	Be											B	C	N	O	F	Ne
Na	Mg											Al	Si	P	S	Cl	Ar
K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn	Sb	Te	I	Xe
Cs	Ba	La	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg	Tl	Pb	Bi	Po	At	Rn
Fr	Ra	Ac	Unq	Unp	Unh	Uns	Uno	Une									
Ce	Pr	Nd	Pm	Sm	Eu	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu				
Th	Pa	U	Np	Pu	Am	Cm	Bk	Cf	Es	Fm	Md	No	Lr				

POISSON EQUATION. A generalization of the Laplace equation is the **Poisson equation** $\Delta u = \rho$, where ρ is a scalar function. If ρ is a **charge distribution**, then u is the **electric potential**: The vector field $E = \text{grad}(u)$ satisfies the Maxwell equation $\text{div}(E) = \rho$. The trick to solve this Maxwell equation is to solve the Poisson equation.

There are solution formulas using **Green's functions** g_d, depending on the dimension d:

1D	$u(x) = 1/2 \int_R \rho(s)\|x-s\| \, ds.$	$g_1 \star \rho$
2D	$u(x) = \frac{-1}{2\pi} \int\int_{R^2} \rho(s) \log\|x-s\| \, dA(s)$	$g_2 \star \rho$
3D	$u(x) = \frac{-1}{4\pi} \int\int\int_{R^3} \rho(s)/\|x-s\| \, dV(s)$	$g_3 \star \rho$

On the right, the notation $f \star b = \int f(x-y)g(y) \, dy$ is used. $*$ is a "multiplication" of functions called **convolution**. Fourier theory allow to prove these formulas with no effort: the Poisson equation $\Delta u = \rho$ becomes after Fourier transform $\sum_{i=1}^d x_i^2 \hat{u} = \hat{\rho}$ so that $\hat{u} = \hat{g}_d \hat{\rho}$, where $\hat{g}_d = 1/\sum_{i=1}^d x_i^2$. Reversing the Fourier transform gives $u = g_d \star \rho$ (there is a general rule $\hat{fg} = \hat{f} \star \hat{g}$.
Because the Fourier transform of $1/x^2$ is $g_1(x) = |x|$, the Fourier transform of $1/(x^2+y^2)$ is $g_2(x) = \frac{1}{4\pi} \log|x^2+y^2|$ and the Fourier transform of $1/(x^2+y^2+z^2)$ is $\frac{1}{2\pi} 1/\sqrt{(x^2+y^2+z^2)}$, the solution formulas become obvious.

UNIQUENESS OF SOLUTIONS? Note that if you take a solution u of the Poisson equation and add a harmonic function u satisfying $\Delta u = 0$, you get an other solution.

ABOUT THE GREEN FUNCTIONS. The Green functions g_d are the natural **Newton potentials** in $\mathbf{R}^d$. The one dimensional potential is studied because two infinite planes attract each other with force $|x|$. The n-body problem with such potentials attractive (gravitational) or repelling (electric) is studied a lot.

EXISTENCE OF SOLUTIONS. Having explicit solutions to partial differential equations is rather exceptional. In general one has to work hard to establish **existence of solutions**. Existence can be subtle. Even for one of the simplest nonlinear PDE's $\text{grad}(u) = F$, there are not necessarily solutions. Only if $\text{curl}(F) = 0$, then a potential u can exist.
For nonlinear PDE's, existence can be very hard to prove. One million dollars have been written out by the Clay institute for a proof in the case of the Navier Stokes equation.

Anyhow, as the following story shows, mathematicians are crazy about existence:

An engineer, a chemist and a mathematician are staying in three adjoining cabins at an old motel.
First the engineer's coffee maker catches fire. He smells the smoke, wakes up, unplugs the coffee maker, throws it out the window, and goes back to sleep.

Later that night the chemist smells smoke too. She wakes up and sees that a cigarette butt has set the trash can on fire. She says to herself, "Hmm. How does one put out a fire? One can reduce the temperature of the fuel below the flash point, isolate the burning material from oxygen, or both. This could be accomplished by applying water." So she picks up the trash can, puts it in the shower stall, turns on the water, and, when the fire is out, goes back to sleep.

The mathematician, of course, has been watching all this out the window. So later, when he finds that his pipe ashes have set the bed-sheet on fire, he is not in the least taken aback. He says:

"Aha! A solution exists!"

and goes back to sleep ...