VECTORS. Vectors $v = (x, y, z)$, the dot product $v \cdot w$ or the vector product $v \times w$.

```
{3,4,5}+2.0*{-4,5,6}            Cross[{3,4,5},{-4,5,6}]
{3,4,5}.{-4,5,6}                l[v_]:=Sqrt[v.v]; l[{3,4,5}]
```

PROJECTION of v onto w is $w(v \cdot w)/|w|^2$.
SCALAR PROJECTION: length of projection.

```
l[v_]:=Sqrt[v.v]; proj[v_,w_]:=w (v.w)/(w.w);
scalarproj[v_,w_]:=l[proj[v,w]];
v0={3,4,5}; w0={4,5,1}; scalarproj[v0,w0]
```

DISTANCES.

$d(P, Q) = |P - Q|$

$d(P, Q + tv) = |(P - Q) \times v)|/|v|$

$d(P, Q + tv + sw) = \frac{|(P-Q)\cdot(v \times w)|}{|v \times w|}$

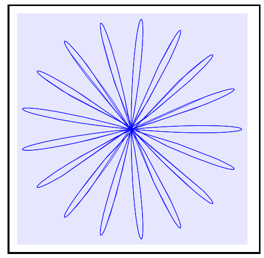
$d(P + tv, Q + sw) = \frac{|(P-Q)\cdot(v \times w)|}{|v \times w|}$

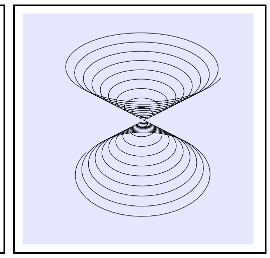
```
P0={2,3,4}; Q0={3,4,6}; v0={1,1,1}; w0={2,4,-1};
l[v_]:=Sqrt[v.v];

PP[P_,Q_]:=Sqrt[(P-Q).(P-Q)];
PL[P_,{Q_,v_}]:=Module[{n=Cross[P-Q,v]/l[v]},l[n]];
PS[P_,{Q_,v_,w_}]:=Module[{n=Cross[v,w]},(P-Q).n/l[n]];
LL[{P_,v_},{Q_,w_}]:=Module[{n=Cross[v,w]},m=l[(P-Q).n]/l[n];

PP[P0,Q0]                       PL[P0,{Q0,v0}]
PS[P0,{Q0,v0,w0}]               LL[{P0,v0},{Q0,w0}]
```

PLANE CURVES. $r(t) = (x(t), y(t))$.
SPACE CURVES. $r(t) = (x(t), y(t), z(t))$.

```
ParametricPlot[{Cos[t] Cos[17t],Sin[t] Cos[17t]},{t,0,2Pi}]
```

```
ParametricPlot3D[{t Cos[10t],t Sin[10t],t},{t,-2Pi,2Pi}]
```

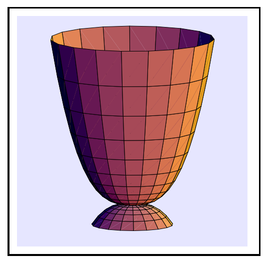


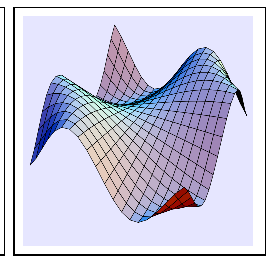


SURFACES. $X(u, v) = (x(u, v), y(u, v), z(u, v))$.
GRAPHS. Graph of map $z = f(x, y)$.

```
ParametricPlot3D[{v Cos[u],v Sin[u],5 v^3},{u,0,2 Pi},{v,-5,10}]
```

```
S=Plot3D[Sin[x*y],{x,-2,2},{y,-2,2}]
```





INTEGRAL. Antiderivative $\int f(x)\, dx$ and definite integral $\int_a^b f(x)\, dx$.

```
Integrate[Cos[Sqrt[x]],x]            NIntegrate[Cos[Sqrt[x]]/x^2,{x,1,Infinity}]
```

LENGTH OF CURVE. $\int_a^b \sqrt{\dot{x} + \dot{y}^2 + \dot{z}^2}\, dt$.
Explicit solutions with "Integrate".

```
r[t_]:={Cos[t],Sin[t],t}; v[t_]:=r'[s] /. s->t;
NIntegrate[Sqrt[v[t].v[t]],{t,0,2Pi}]
```

DOUBLE INTEGRAL. $\int_a^b \int_c^d f(x, y)\, dxdy$.

```
Integrate[ Sin[x] x^2+y^3,{x,0,2 Pi},{y,0,3 Pi} ]
```

TRIPLE INTEGRAL. $\int_a^b \int_c^d \int_e^f f(x, y, z)\, dxdydz$.

```
Integrate[ Sin[x] x^2+y^3+z^4,{x,0,2 Pi},{y,0,3 Pi},{z,0,Pi} ]
```

LINE INTEGRAL. $\int_a^b F(r(t)) \cdot r'(t)\, dt$.

```
r[t_]:={Cos[t],Sin[t],t}; v[t_]:=r'[s] /. s->t;
F[{x_,y_,z_}]:={x^2,y,z}; NIntegrate[F[r[t]].v[t],{t,0,2Pi}]
```

FLUX INTEGRAL.
$\int_a^b \int_c^d F(u, v) \cdot (X_u \times X_v)(u, v)\, dvdu$.

```
X[u_,v_]:={Cos[u],Sin[u],v}; F[{x_,y_,z_}]:={x^2,y,z};
Xu[u_,v_]:=D[X[s,v],s] /. s->u; Xv[u_,v_]:=D[X[u,t],t] /. t->v;
R[u_,v_]:=Cross[Xu[u,v],Xv[u,v]];
Integrate[F[X[u,v]].R[u,v],{u,0,2Pi},{v,0,1}]
```

LAGRANGE MULTIPLIERS. Extremize $F(x, y, z)$ under constraint $G(x, y, z) = c$. Solve system $\nabla F(x, y, z) = \lambda \nabla G(x, y, z), G(x, y, z) = c$ for the x, y, z, λ.

```
F[x_,y_,z_]:=-x^2-y^2-z^2;
G[x_,y_,z_]:=x+y+z;
Solve[{D[F[x,y,z],x]== L*D[G[x,y,z],x],
       D[F[x,y,z],y]== L*D[G[x,y,z],y],
       D[F[x,y,z],z]== L*D[G[x,y,z],z],
       G[x,y,z]==1},{x,y,z,L}]
```

GRAD CURL DIV.
$\mathrm{grad}(f) = (f_x, f_y, f_z)$
$\mathrm{rurl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$
$\mathrm{div}(P, Q, R) = P_x + Q_y + R_z$.

```
f[x_,y_,z_]:=x^2 y z; F[x_,y_,z_]:={-y,x,z^2 x};
grad[U_]:={D[U[x,y,z],x],D[U[x,y,z],y],D[U[x,y,z],z]};
curl[F_]:={D[F[x,y,z][[3]],y]-D[F[x,y,z][[2]],z],
           D[F[x,y,z][[1]],z]-D[F[x,y,z][[1]],x],
           D[F[x,y,z][[2]],x]-D[F[x,y,z][[3]],y]};
div[F_]:=D[F[x,y,z][[1]],x]
        +D[F[x,y,z][[2]],y]
        +D[F[x,y,z][[3]],z];
```

LINEAR APPROXIMATION.
$L(x, y, z) = f(x, y, z) + \mathrm{grad}(f)(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$

```
f[x_,y_,z_]:=Sin[x*y]*z-x^2*y;
Dx[f_][x_,y_,z_]:=D[f[u,v,w],u] /. {u->x,v->y,w->z}
Dy[f_][x_,y_,z_]:=D[f[u,v,w],v] /. {u->x,v->y,w->z}
Dz[f_][x_,y_,z_]:=D[f[u,v,w],w] /. {u->x,v->y,w->z}
grad[f_][x_,y_,z_]:={Dx[f][x,y,z],Dy[f][x,y,z],Dz[f][x,y,z]};
L[x_,y_,z_]:=f[1,1,1]+grad[f][1,1,1].{x-1,y-1,z-1}
```

QUADRATIC APPROXIMATION.
$Q(x, y) = L(x, y) + [f''(x_0, y_0)(x - x_0, y - y_0)] \cdot (x - x_0, y - y_0)/2$.

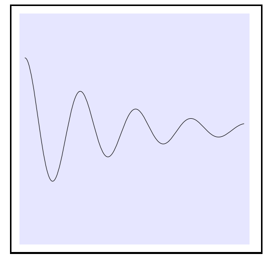
```
f[x_,y_]:=Sin[x*y]-x^2*y;
Dx[f_][x_,y_]:=D[f[u,v],u] /. {u->x,v->y}
Dy[f_][x_,y_]:=D[f[u,v],v] /. {u->x,v->y}
grad[f_][x_,y_]:={Dx[f][x,y,z],Dy[f][x,y,z]};
Dxx[f_][x_,y_]:=D[f[u,v],{u,2}]  /. {u->x,v->y}
Dxy[f_][x_,y_]:=D[D[f[u,v],u],v] /. {u->x,v->y}
Dyy[f_][x_,y_]:=D[f[u,v],{v,2}]  /. {u->x,v->y}
hess[f_][x_,y_]:={{Dxx[f][x,y],Dxy[f][x,y]},{Dxy[f][x,y],Dyy[f][x,y]}}
Q[x_,y_]:=f[1,1]+grad[f][1,1].{x-1,y-1}+(hess[f][1,1].{x-1,y-1}).{x-1,y-1}/2
```

ODE's. $p'[t] = f(p, t)$

```
DSolve[D[p[x],x]==p[x]*(1-p[x]),p[x],x]
```

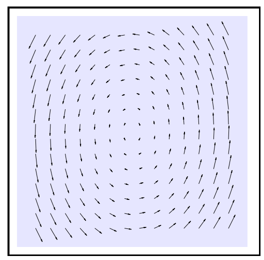
INITIAL VALUE PROBLEM. $x'' = x - \mu x', x[0] = a, x'[0] = b$

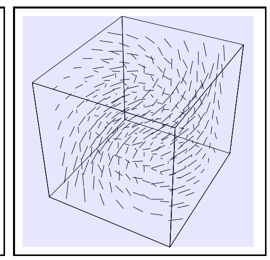
```
U=DSolve[{x''[t]==-x[t]-x'[t]/5,x[0]==1,x'[0]==0},{x},t];
S=Plot[Evaluate[x[t] /. U],{t,0,25}]
```



VECTOR FIELD. $F(x, y) = (P(x, y), Q(x, y))$

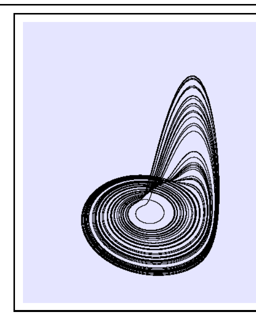
```
Needs["Graphics`PlotField`"]
PlotVectorField[{-y,2 x},{x,-1,1},{y,-1,1}]
PlotGradientField[x^2 y,{x,-1,1},{y,-1,1}]

Needs["Graphics`PlotField3D`"]
PlotVectorField3D[{-y,2 x,z},{x,-1,1},{y,-1,1},{z,-1,1}]
PlotGradientField3D[x^2 y z,{x,-1,1},{y,-1,1},{z,-1,1}]
```





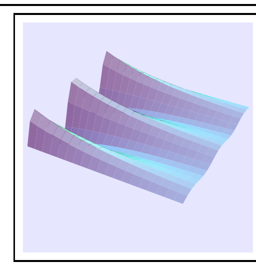
ODE's in space $\dot{x} = F(x)$. The example shows the Rössler attractor.

```
s=NDSolve[{x'[t]==-(y[t]+z[t]),y'[t]==x[t] +0.2*y[t],z'[t]==.2+x[t]*z[t]-5.7*z[t],
x[0]==.3,z[0]==4.2,y[0]==1.3},{x,y,z},{t,0,400},MaxSteps->12000];
S=ParametricPlot3D[Evaluate[{x[t],y[t],z[t]}/.s],{t,0,400}]
```



HEAT EQUATION $u_t = \mu u_{xx}$.

```
s=NDSolve[{D[u[x,t],t]==D[u[x,t],{x,2}],
    u[x,0]==Sin[5*Pi*x],u[0,t]==0, u[1,t]==0},u,{x,0,1},{t,0,.01}];
Plot3D[Evaluate[u[x,t]/. s[[1]]],{t,0,.01},{x,0,1}]
```



WAVE EQUATION $u_{tt} = c^2 u_{xx}$.

```
s=NDSolve[{D[u[x,t],{t,2}] == D[u[x,t],{x,2}],
         u[x,0] == Sin[4*Pi*x], Derivative[0,1][u][x,0]==0,
         u[0,t] == u[1,t]}, u, {x,0,1}, {t,0,1}];
Plot3D[Evaluate[u[x,t]/. s[[1]]],{t,0,1},{x,0,1}]
```