SUMMARY. We review the fundamental theorem of line integrals, Green's theorem, Stokes theorem and Gauss theorem. These four results generalize the fundamental theorem of calculus and are all of the form $\boxed{\int_R dF = \int_{dR} F}$, where dR is the boundary of R and dF is a derivative of F.

INTEGRATION.

Line integral: $\int_\gamma F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\ dt$

Surface integral $\int\int_S f dS = \int_a^b \int_c^d f(X(u,v))|X_u(u,v) \times X_v(u,v)|\ dudv$

Flux integral: $\int\int_S F \cdot dS = \int_a^b \int_c^d F(X(u,v)) \cdot X_u(u,v) \times X_v(u,v)\ dudv$

Double integral: $\int\int_R f\ dA = \int_a^b \int_c^d f(x,y)\ dxdy.$

Triple integral: $\int\int\int_R f\ dV = \int_a^b \int_c^d \int_o^p f(x,y,z)\ dxdydz.$

Area $\int\int_R 1dA = \int\int_R 1\ dxdy$

Surface area $\int\int 1dS = \int\int |X_u \times X_v| dudv$

Volume $\int\int\int_B 1\ dV = \int\int\int_B 1\ dxdydz$

DIFFERENTIATION.

Derivative: $f'(t) = \dot{f}(t) = d/dt f(t).$

Partial derivative: $U_x(x,y,z) = \frac{\partial U}{\partial x}(x,y,z).$

Gradient: $\text{grad}(U) = (U_x, U_y, U_z)$

Curl in 2D: $\text{curl}(F) = \text{curl}((P,Q)) = Q_x - P_y$

Curl in 3D: $\text{curl}(F) = \text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$

Div: $\text{div}(F) = \text{div}(P,Q,R) = P_x + Q_y + R_z.$

IDENTITIES.
$\text{div}(\text{curl}(F)) = 0$
$\text{curl}(\text{grad}(U)) = (0,0,0)$

LINE INTEGRAL THEOREM. If $\gamma : t \mapsto r(t), t \in [a,b]$ is a curve and U is a function either in 3D or the plane. Then

$$\boxed{\int_\gamma \nabla U \cdot ds = r(b) - r(a)}$$

where $\nabla U = \text{grad}(U)$ is the gradient of U.

CONSEQUENCES.
1) If the curve is closed, then the line integral $\int_\gamma \nabla U \cdot ds$ is zero.
2) If $F = \nabla U$, the line integral between two points P and Q does not depend on the chosen path.

REMARKS.

1) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus**

$$\boxed{\int_a^b U'(x)\ dx = U(b) - U(a)}$$

2) The theorem justifies the name **conservative** for gradient vector fields.
3) In physics, U is the **potential energy** and ∇U a **force**. The theorem says that for such forces we have **energy conservation**.

PROBLEM. Let $U(x,y,z) = x^2 + y^4 + z$. Find the line integral of the vector field $F(x,y,z) = \nabla U(x,y,z)$ along the path $r(t) = (\cos(5t), \sin(2t), t^2)$ from $t = 0$ to $t = 2\pi$.

SOLUTION. $r(0) = (1,0,0)$ and $r(2\pi) = (1,0,4\pi^2)$ and $U(r(0)) = 1$ and $U(r(2\pi)) = 1 + 4\pi^2$. Applying the fundamental theorem gives $\int_\gamma \nabla U\ ds = U(r(2\pi)) - U(r(0)) = 4\pi^2$.

GREEN'S THEOREM. If R is a region with boundary γ and $F = (P,Q)$ is a vector field, then

$$\boxed{\int\int_R \text{curl}(F)\ dA = \int_\gamma F \cdot ds}$$

where $\text{curl}(F)(x,y) = Q_x - P_y$.

REMARKS.
1) The theorem can be used to calculate two dimensional integrals by a one dimensional integral along a curve. Sometimes, however, it is used to evaluate line integrals by evaluating 2D integrals.
2) The curve is oriented in such a way that the region is to your left.
3) The region has to has piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set).
4) If $\gamma : t \mapsto r(t) = (x(t), y(t))$, the line integral is $\int_a^b (P(x(t),y(t)), Q(x(t),y(t)) \cdot (x'(t), y'(t))\ dt.$
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Michel Ostogradski (1801-1861).

CONSEQUENCES.
1) If $\text{curl}(F) = 0$ everywhere in space or the plane, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
2) Taking $F(x, y) = (-y, 0)$ gives an **area formula** $\text{Area}(R) = \int -y \, dx$. Similarly $\text{Area}(A) = \int x dy$.

PROBLEM. Find the line integral of the vector field $F(x, y) = (x^4 + \sin(x) + y, x + y^3)$ along the path $r(t) = (\cos(t), 5\sin(t) + \log(1 + \sin(t)))$, where t runs from $t = 0$ to $t = \pi$.

SOLUTION. $\text{curl}(F) = 0$ imples that the line integral depends only on the end points $(0, 1), (0, -1)$ of the path. Take the simpler path $r(t) = (-t, 0), t = [-1, 1]$, which has velocity $r'(t) = (-1, 0)$. The line integral is $\int_{-1}^{1} (t^4 - \sin(t), -t) \cdot (-1, 0) \, dt = -t^5/5|_{-1}^{1} = -2/5$.

REMARK. We could also find a potential $U(x, y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(U) = F$. Again, we get $U(0, -1) - U(0, 1) = -1/5 - 1/5 = -2/5$.

STOKES THEOREM. If S is a surface in space with boundary γ and F is a vector field, then

$$\int\int_S \text{curl}(F) \cdot dS = \int_\gamma F \cdot ds$$

REMARKS.
1) Stokes theorem reduces to Greens theorem if F is z independent and S is contained in the z-plane.
2) The orientation of γ is such that if you walk along γ and have your head in the direction, where the normal vector $X_u \times X_v$ of S points, then you have the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825. It was rediscovered by George Stokes (1819-1903).

CONSEQUENCES.
1) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S. This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
2) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

PROBLEM. Compute the line integral of $F(x, y, z) = (x^3 + xy, y, z)$ along the polygonal path γ connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.

SOLUTION. The path γ bounds a surface $S : X(u, v) = (u, v, 0)$ parameterized by $R = [0, 2] \times [0, 1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(F)(x, y, z) = (0, 0, -x)$ through S. The normal vector of S is $X_u \times X_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$ so that $\int\int_S \text{curl}(F) \, dS = \int_0^2 \int_0^1 (0, 0, -u) \cdot (0, 0, 1) \, dudv = \int_0^2 \int_0^1 -u \, dudv = -2$.

GAUSS THEOREM. If S is the boundary of a region B in space with boundary S and F is a vector field, then

$$\int\int\int_B \text{div}(F) \, dV = \int\int_S F \cdot dS$$

where $\text{div}(F)$ is the divergence of F.

REMARKS.
1) Gauss theorem is also called **divergence theorem**.
2) Gauss theorem is helpful to determine the flux of vector fields through surfaces.
3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
CONSEQUENCES.
1) For divergence free vector fields F, the flux through a closed surface is zero. Such fields F are also called **incompressible** or **source free**.

PROBLEM. Compute the flux of the vector field $F(x, y, z) = (-x, y, z^2)$ through the boundary S of the rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.

SOLUTION. By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \, dxdydz = (3 - 0)(2 - (-1))(4 - 1) = 27$.