CROSS PRODUCT. The **cross product** of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined as $v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$.

AREA. $v \times w$ is orthogonal to v and orthogonal to w. Its length $|v \times w|$ is the area of parallelogram spanned by v and w. Proof. Check it first for $v = (1, 0, 0)$ and $w = (\cos(\alpha), \sin(\alpha), 0)$, where $v \times w = (0, 0, \sin(\alpha))$ has length $|\sin(\alpha)|$ which is indeed the area of the parallelogram spanned by v and w. A more general case can be obtained by scaling v and w: both the area as well as the cross product behave linearly in v and w.

The formula

$$|v \times w| = |v||w| \sin(\alpha)$$

(which can be checked also using $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$ and $|v \cdot w| = |v||w| \cos(\alpha)$, gives an other way to measure angles. We see that $v \times w$ is zero if v and w are parallel or one of the vectors is zero.

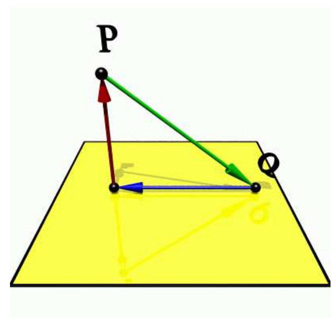
DOT PRODUCT (is scalar)		CROSS PRODUCT (is vector)	
$v \cdot w = w \cdot v$	commutative	$v \times w = -w \times v$	anti-commutative
$\|v \cdot w\| = \|v\|\|w\| \cos(\alpha)$	angle	$\|v \times w\| = \|v\|\|w\| \sin(\alpha)$	angle
$(av) \cdot w = a(v \cdot w)$	linearity	$(av) \times w = a(v \times w)$	linearity
$(u + v) \cdot w = u \cdot w + v \cdot w$	distributivity	$(u + v) \times w = u \times w + v \times w$	distributivity
$\{1, 2, 3\}.\{3, 4, 5\}$	in Mathematica	Cross$[\{1, 2, 3\}, \{3, 4, 5\}]$	Mathematica
$\frac{d}{dt}(v \cdot w) = \dot{v} \cdot w + v \cdot \dot{w}$	product rule	$\frac{d}{dt}(v \times w) = \dot{v} \times w + v \times \dot{w}$	product rule

TRIPLE SCALAR PRODUCT. The scalars $[u, v, w] = u \cdot v \times w$ is called the **triple scalar product** of u, v, w. It is the volume of the parallelepiped spanned by u, v, w because $u \cdot n$ is the height of the parallelepiped if n is a normal vector to the ground parallelogram which has area $|v \times w|$.

DISTANCE POINT-PLANE (3D). If P is a point in space and $n \cdot x = d$ is a plane containing a point Q, then

$$d(P, L) = |(P - Q) \cdot n|/|n|$$

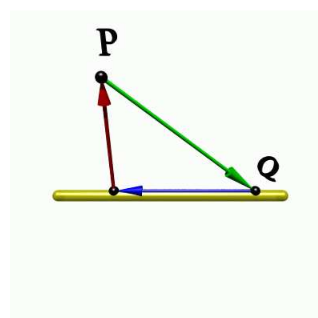
is the distance between P and the plane.



DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $r(t) = Q + tu$, then

$$d(P, L) = |(P - Q) \times u|/|u|$$

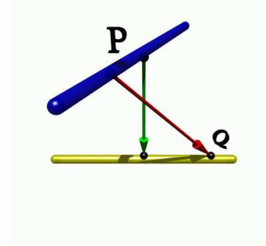
is the distance between P and the line L.



DISTANCE LINE-LINE (3D). L is the line $r(t) = Q + tu$ and M is the line $s(t) = P + tv$, then

$$d(L, M) = |(P - Q) \cdot (u \times v)|/|u \times v|$$

is the distance between the two lines L and M.



PLANE THROUGH 3 POINTS P, Q, R: The vector $(a, b, c) = n = (Q - P) \times (R - P)$ is normal to the plane. Therefore, the equation is $ax + by + cz = d$. The constant is $d = ax_0 + by_0 + cz_0$ because $P = (x_0, y_0, z_0)$ must be on the plane.

PLANE THROUGH POINT P AND LINE $r(t) = Q + tu$. The vector $(a, b, c) = n = u \times (Q - P)$ is normal to the plane. Therefore the plane is given by $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$ and $P = (x_0, y_0, z_0)$.

LINE ORTHOGONAL TO PLANE ax+by+cz=d THROUGH POINT P. The vector $n = (a, b, c)$ is normal to the plane. The line is $r(t) = P + nt$.

ANGLE BETWEEN PLANES. The angle between the two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $\arccos(\frac{n_1 \cdot n_2}{|n_1||n_2|}$, where $n_i = (a_i, b_i, c_i)$. Alternatively, it is $\arcsin(\frac{n_1 \times n_2}{|n_1||n_2|}$.

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$. Find first a point P which is in the intersection. Then $r(t) = P + t(n_1 \times n_2)$ is the line, we were looking for.

ANGULAR MOMENTUM. If a mass point of mass m moves along a curve $r(t)$, then the vector $L(t) = mr(t) \times r'(t)$ is called the **angular momentum**.

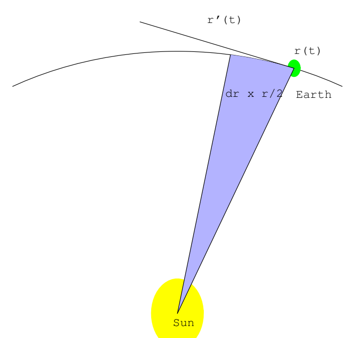
ANGULAR MOMENTUM CONSERVATION.

$$\frac{d}{dt}L(t) = mr'(t) \times r'(t) + mr(t) \times r''(t) = r(t) \times F(t)$$

In a central field, where $F(t)$ is parallel to $r(t)$, this vanishes.

TORQUE. The quantity $r(t) \times F(t)$ is called the **torque**. The time derivative of the **momentum** mr' is the **force**, the time derivative of the **angular momentum** L is the **torque**.

KEPLER'S AREA LAW. (Proof by Newton) The fact that $L(t)$ is constant means first of all that $r(t)$ stays in a plane spanned by $r(0)$ and $r'(0)$. The experimental fact that the vector $r(t)$ sweeps over **equal areas in equal times** expresses the angular momentum conservation: $|r(t) \times r'(t)dt/2| = |Ldt/m/2|$ is the area of a small triangle. The vector $r(t)$ sweeps over an area $\int_0^T Ldt/(2m) = LT/(2m)$ in time $[0, T]$.



PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS: (informal)
In a rotating coordinate system a particle of mass m moving along $r(t)$ experience the following forces: $m\omega' \times r$ (inertia of rotation), $2m\omega \times r'$ (Coriolis force) and $m\omega \times (\omega \times r))$ (Centrifugal force).

The **top**, the motion of a rigid body is describe by the angular momentum M and the angular velocity vector Ω in the body. Then $\dot{M} = M \times \Omega + F$, where F is an external force.

Electromagnetism: a particle moving along $r(t)$ in a **magnetic field** B for example experiences the force $F(t) = qr'(t) \times B$, where q is the charge of the particle.