DOT PRODUCT. The **dot product** of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined as $v \cdot w = v_1w_1 + v_2w_2 + v_3w_3$. Other notations are $v \cdot w = (v, w)$ or $< v|w >$ (quantum mechanics) or $v_i w^i$ (Einstein notation) or $g_{ij}v^i w^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

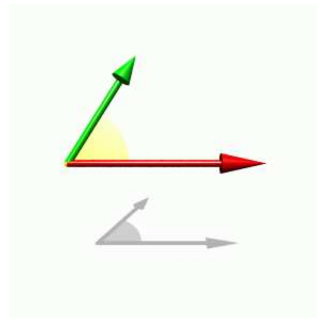
LENGTH. Using the dot product one can express the length of v as $||v|| = \sqrt{v \cdot v}$.

The dot product can be recovered from the length only because $(v + w, v + w) = (v, v) + (w, w) + 2(v, w)$ can be solved for (v, w).

ANGLE. Because $||v - w||^2 = (v - w, v - w) = ||v||^2 + ||w||^2 - 2(v, w)$ is by the **cos-theorem** equal to $||v||^2 + ||w||^2 - 2||v|| \cdot ||w|| \cos(\phi)$, where ϕ is the angle between the vectors v and w, we have the formula

$$v \cdot w = ||v|| \cdot ||w|| \cos(\phi) \; .$$

Using the dot product, we can measure angles!



FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.
ANSWER: $\cos(\phi) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\phi = \arccos(0.839..) \sim 33°$.

ORTHOGONALITY. Two vectors are called **orthogonal** if $v \cdot w = 0$.

PROJECTION. The vector $a = w(v \cdot w/||w||^2)$ is called the **projection** of v onto w. Its length $||a|| = (v \cdot w/||w||^2)$ is called the **scalar projection**. The vector $b = v - a$ is the **component** of v orthogonal to the w-direction.

PLANES. A surface $ax + by + cz = d$ in space is called a **plane**. Using the dot product we can say $n \cdot x = d$, where $n = (a, b, c)$ and $x = (x, y, z)$.

NORMAL VECTOR If x, y are two points on the plane, then $v \cdot x = d$ and $v \cdot y = d$ which means $v \cdot (x - y) = 0$. Every vector in the plane is orthogonal to v. The later called the **normal vector** to the plane.

EXAMPLE, WHERE DOT PRODUCT APPEARS IN PHYSICS (informal)

ENERGY. $K(t) = m||r'(t)||^2/2$ is called the **kinetic energy** of a body of mass m moving along a path $r(t)$.

WORK. If a body moves with speed $r'(t)$ and is exposed to a force $F(t)$ then it gains the work energy $W = \int_a^b r'(t) \cdot F(t) \; dt$.

ENERGY CONSERVATION. If the energy $K(t)$ is differentiated with respect to time, we get $K'(t) = m(r''(t) \cdot r'(t))$. By Newton's law $mr''(t) = F(t)$ and using the fundamental theorem of calculus, we have

$$K(b) - K(a) = \int_a^b K'(t) \; dt = \int_a^b r'(t) \cdot mr''(t) \; dt = \int_a^b r'(t) \cdot F(t) \; dt = W$$

Energy conservation: the kinetic energy difference is the amount of work which has been used.

EXAMPLE, WHERE DOT PRODUCTS APPEAR IN GEOMETRY.
The distance between a point P and a plane $ax + by + cz = d$ is obtained by taking a point Q on the plane and taking the scalar projection of the vector $P - Q$ onto n.

CROSS PRODUCT. The **cross product** of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined as $v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$.

by v and w. Proof: Check it first for $v = (1,0,0)$ and $w = (\cos(\alpha), \sin(\alpha), 0)$, where $v \times w = (0,0,\sin(\alpha))$ has length $|\sin(\alpha)|$ which is indeed the area of the parallelogram spanned by v and w. A more general case can be obtained by scaling v and w: both the area as well as the cross product behave linearly in v and w.

The formula

$$|v \times w| = |v||w|\sin(\alpha)$$

(which can be checked also using $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$ and $|v \cdot w| = |v||w|\cos(\alpha)$, gives a different way to measure angles. We see that $v \times w$ is zero if v and w are parallel or one of the vectors is zero.

Here is a overview of properties of the dot product and cross product.

DOT PRODUCT (is scalar)		CROSS PRODUCT (is vector)	
$v \cdot w = w \cdot v$	commutative	$v \times w = -w \times v$	anti-commutative
$\|v \cdot w\| = \|v\|\|w\|\cos(\alpha)$	angle	$\|v \times w\| = \|v\|\|w\|\sin(\alpha)$	angle
$(av) \cdot w = a(v \cdot w)$	linearity	$(av) \times w = a(v \times w)$	linearity
$(u+v) \cdot w = u \cdot w + v \cdot w$	distributivity	$(u+v) \times w = u \times w + v \times w$	distributivity
$\{1,2,3\}.\{3,4,5\}$	in Mathematica	Cross[$\{1,2,3\}, \{3,4,5\}$]	Mathematica
$\frac{d}{dt}(v \cdot w) = \dot{v} \cdot w + v \cdot \dot{w}$	product rule	$\frac{d}{dt}(v \times w) = \dot{v} \times w + v \times \dot{w}$	product rule

TRIPLE SCALAR PRODUCT. The scalars $[u,v,w] = u \cdot v \times w$ is called the **triple scalar product** of u, v, w. It is the volume of the parallelepiped spanned by u, v, w because $u \cdot n$ is the height of the parallelepiped if n is a normal vector to the ground parallelogram which has area $|v \times w|$.

PARAMETERIZED PLANE. A plane can also be given by a point P and two vectors v, w. Just look at all points $P + tv + sw$, where t and s are real numbers. How do we obtain an equation $ax + by + cz = 0$ from this? Answer: the vector $v \times w$ is orthogonal to the plane and therefore a multple of (a,b,c). The number d is equal to $aP_1 + bP_2 + cP_3$ if $P = (P_1, P_2, P_3)$ is the point on the surface.

EXAMPLE. Find the equation $ax + by + dz = d$ for the plane $\{(1,1,1) + s(1,0,1) + t(2,1,0) | s,t \in \mathbf{R}\ \}$. Solution: $n = (1,0,1) \times (2,1,0) = (-1,2,1)$. The equation of the plane is therefore $-x + 2y + z = d$. We get $d = n \cdot P = (-1,2,1) \cdot (1,1,1) = 2$. Result: $-x + 2y + z = 2$.

PLANE THROUGH 3 POINTS P, Q, R: The vector $(a,b,c) = n = (Q - P) \times (R - P)$ is normal to the plane. Therefore, the equation is $ax + by + cz = d$. The constant is $d = ax_0 + by_0 + cz_0$ because $P = (x_0, y_0, z_0)$ must be on the plane.

PLANE THROUGH POINT P AND LINE $r(t) = Q + tu$. The vector $(a,b,c) = n = u \times (Q - P)$ is normal to the plane. Therefore the plane is given by $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$ and $P = (x_0, y_0, z_0)$.

LINE ORTHOGONAL TO PLANE ax+by+cz=d THROUGH THE POINT P. The vector $n = (a,b,c)$ is normal to the plane. The line is $r(t) = P + nt$.

ANGLE BETWEEN PLANES. The angle between the two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $\arccos(\frac{n_1 \cdot n_2}{|n_1||n_2|}$, where $n_i = (a_i, b_i, c_i)$. Alternatively, it is $\arcsin(\frac{n_1 \times n_2}{|n_1||n_2|}$.

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$. Find first a point P which is in the intersection. Then $r(t) = P + t(n_1 \times n_2)$ is the line, we were looking for.

PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS: (informal)

In a rotating coordinate system a particle of mass m moving along $r(t)$ experience the following forces: $m\omega' \times r$ (inertia of rotation), $2m\omega \times r'$ (Coriolis force) and $m\omega \times (\omega \times r))$ (Centrifugal force).

The **top**, the motion of a rigid body is describe by the angular momentum M and the angular velocity vector Ω in the body. Then $\dot{M} = M \times \Omega + F$, where F is an external force.

Electromagnetism: a particle moving along $r(t)$ in a **magnetic field** B for example experiences the force $F(t) = qr'(t) \times B$, where q is the charge of the particle.