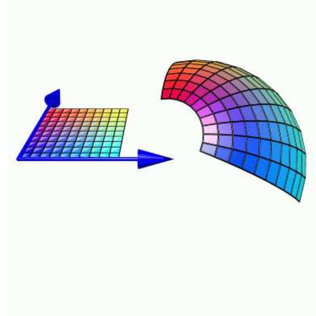
PARAMETRIZED SURFACES. The image of a map $X(u,v) = (x(u,v), y(u,v), z(u,v))$ is a **surface**. X is called the parameterization of the surface. To define the surface, we have to be given three functions $x(u,v), y(u,v), z(u,v)$ of two variables. If we fix one of the variables, say $v = v_0$, then $u \mapsto X(u, v_0)$ is a curve on the surface. Similarly, if we fix $u = u_0$, then $v \mapsto X(u_0, v)$ is a curve on the surface.

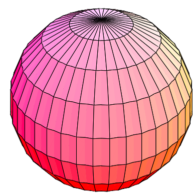


TWO WAYS TO REPRESENT A SURFACE.

- I) Solutions of an equation $g(x,y,z) = 0$.
- II) Parameterizations as image of $X(u,v)$.

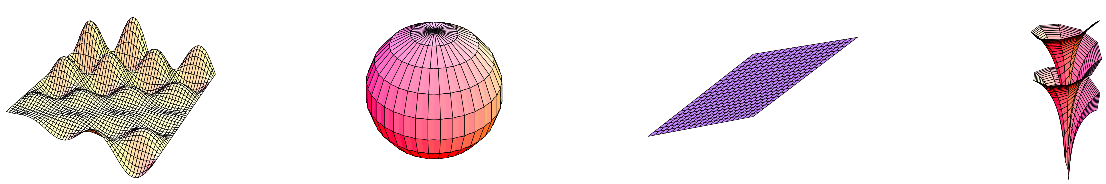
EXAMPLE GRAPH. The graph of a function $f(x,y)$ can be parametrized as $(u,v) \mapsto X(u,v) = (u, v, f(u,v))$ and can also be written as $\{g(x,y,z) = z - f(x,y) = 0\}$.

EXAMPLE SPHERE. The sphere is the set of (x,y,z) for which $g(x,y,z) = x^2 + y^2 + z^2 = 1$. The sphere is also the image of the parametrisation $X(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$. (The upper half sphere is the graph of the function $f(x,y) = \sqrt{1 - x^2 - y^2}$. We could also write the sphere as a graph of $\rho(\theta,\phi) = r$ in spherical coordinates).



NOTE. There are surfaces, which can also not represented as an image of a single parameterisation X. In general, one needs different patches. Mathematicians make this precise with the notion of a "manifold".

EXAMPLES. of parameterized surface $(u,v) \mapsto X(u,v)$



Graphs	Sphere	Plane through points P, Q, R	Dini surface
$\begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$	$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$	$P+$ $u(Q-P)+$ $v(R-P)$	$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix}$

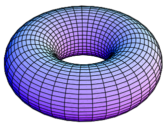
SURFACE OF REVOLUTION.

When spinning a graph $y = f(x)$ around the x-axes, we obtain a **surface of revolution**. Keeping $u = x$ as one of the parameters and v as the angle of rotation and $f(u)$ as the radius, we get $x(u,v) = u, y(u,v) = f(u)\cos(v), z(u,v) = f(u)\sin(v)$. Therefore, $X(u,v) = (u, f(u)\cos(v), f(u)\sin(v))$.

For example, for $f(x) = x$, we obtain a cone, for $f(x) = x^2$, we obtain a paraboloid.

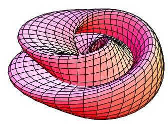
GRID LINES. If we keep u constant, then $v \mapsto X(u,v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto X(u,v)$ is a curve on the surface. These lines are called **grid curves**. If you plot a surface with a computer, the pictures usually show these grid lines.

MORE EXAMPLES. Parameterized surfaces $(u,v) \mapsto X(u,v)$



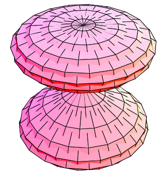
Torus

$$\begin{bmatrix} (2+\cos(v))\cos(u) \\ (2+\cos(v))\sin(u) \\ \sin(v) \end{bmatrix}$$



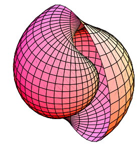
Klein bottle

$$\begin{bmatrix} (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\cos(2u) \\ (\frac{3}{2}+\cos(u)\sin(2v)-\sin(u)\sin(4v))\sin(2u) \\ \frac{3}{4}\sin(u)\sin(2v)+\cos(u)\sin(4v) \end{bmatrix}$$



Eight surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v)+\log(\tan(\frac{v}{2}))+\frac{u}{5} \end{bmatrix}$$

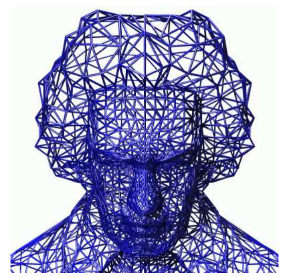


Snail surface

$$\begin{bmatrix} \cos(u)\sin(2v) \\ \sin(u)\sin(2v) \\ \sin(v) \end{bmatrix}$$

WHERE DO SURFACES OCCUR?

Computer graphics. i.e. modeling of faces, terrains, cars, spaceships etc. The map X is called a **u-v map**. Such maps are used to apply textures to three-dimensional models. There are software utilities, which allows an artist to draw directly onto the R domain. The software then applies the map automatically onto the surface. You can paint like this on surfaces in space.



Level surfaces. Level surfaces of functions of three variables. Example, surface of constant temperature in the ocean, surface of constant pressure in the atmosphere.

Graphs. For example: height functions, probability distribution functions.

Intuition Intuition for higher dimensional surfaces (or manifolds) is often obtained from 2-dimensional surfaces in three dimensional space. Higher dimensional surfaces appear everywhere. Our universe for example is modeled as a four dimensional manifold in general relativity. The planetary motion in our the solar system is modeled by a flow on a $9*6-10$ dimensional surface.

D-branes. In super-string theory, surfaces called **Dp-branes** appear. (The letter D stands for "Dirichlet"). D1-branes are called D-strings. D2-branes are surfaces.

Artistic. Last but not least, there is the artistic or esthetic aspect. Examples are paintings by Fomenko, a Russian topologist. An other example is the surface at the entrance of the science center (see photo). You walk by this surface every day.