# FIRST MIDTERM REVIEW — Math 21a

GENERAL TIPS.

- Practice midterm, practice TF questions,
- Make list of facts on a sheet of paper,
- Fresh up short-term memory before test
- Review homework, especially errors. Find error patterns.
- Ask questions:

  "Ask a question and you're a fool for three minutes; do not ask a question and you're a fool for the rest of your life." - Chinese Proverb

- During the exam: read the questions carefully. Wrong understanding makes you do an other job:

  There was a college student trying to earn some pocket money by going from house to house offering to do odd jobs. He explained this to a man who answered one door. "How much will you charge to paint my porch?" asked the man. "Forty dollars." "Fine" said the man, and gave the student the paint and brushes. Three hours later the paint-splattered lad knocked on the door again. "All done!", he says, and collects his money. "By the way," the student says, "That's not a Porsche, it's a Ferrari."

MIDTERM TOPICS.

- Properties of dot, cross and triple product
- Orthogonal, parallel, projection
- Parametrized Lines and Planes
- Switch between parameterization and equations
- Given line and plane, find intersection
- Given plane and plane, find intersection
- Given line and point, find plane
- Given point and point, find line
- Given three points, find plane
- Distance between two points
- Distance between point and plane
- Distance between point and line
- Analyze curves
- Determine curves from acceleration (Boba Fett)
- Distance between two lines
- Distance between two planes
- Angle between two vectors
- Angle between two planes
- Area of parallelogram, triangle in space
- Volume of parallel epiped
- Rectangular/spherical/cylindrical Coordinates
- Identify surfaces in spherical coordinates
- Identify surfaces in cylindrical coordinates
- Distinguish quadrics
- Distinguish graphs
- Traces, intercepts, generalized traces.
- Compute velocity, acceleration
- Find length of curves
- Parameterize curves by intersecting two surfaces

VECTORS.

Two points $P = (1, 2, 3), Q = (3, 4, 6)$ define a vector $\vec{v} = \vec{PQ} = \langle 2, 2, 3\rangle$. If $\vec{v} = \lambda\vec{w}$, then the vectors are **parallel** if $\vec{v} \cdot \vec{w} = 0$, then the vectors are called **orthogonal**. For example, $(1, 2, 3)$ is parallel to $(-2, -4, -6)$ and orthogonal to $(3, -2, 1)$. The addition, subtraction and scalar multiplication of vectors is done componentwise. For example: $(3, 2, 1) - 2((1, 1, 1) + (-1, -1, 0)) = (3, 2, -1)$.

A nonzero vector $\vec{v}$ and a point $P$ define a line $r(t) = P + t\vec{v}$. Two nonzero, nonparallel vectors $\vec{v}, \vec{w}$ and a point $P$ define a plane $P + t\vec{v} + s\vec{s}$. The vector $\vec{n} = \vec{v} \times \vec{w} = (a, b, c)$ is orthogonal to the plane. The points on the plane satisfy an equation $ax+by+cz = d$, where $d$ is obtained by replacing $(x, y, z)$ with a point on the plane. With the help of the dot product for projection and the dot product to get orthogonal vectors, one can solve most geometric problems in 3D.

DOT PRODUCT (is scalar)

| | |
|---|---|
| $v \cdot w = w \cdot v$ | commutative |
| $\|v \cdot w\| = \|v\|\|w\| \cos(\alpha)$ | angle |
| $(av) \cdot w = a(v \cdot w)$ | linearity |
| $(u+v) \cdot w = u \cdot w + v \cdot w$ | distributivity |
| $\{1,2,3\}.\{3,4,5\}$ | in Mathematica |
| $\frac{d}{dt}(v \cdot w) = \dot{v} \cdot w + v \cdot \dot{w}$ | product rule |

CROSS PRODUCT (is vector)

| | |
|---|---|
| $v \times w = -w \times v$ | anti-commutative |
| $\|v \times w\| = \|v\|\|w\| \sin(\alpha)$ | angle |
| $(av) \times w = a(v \times w)$ | linearity |
| $(u+v) \times w = u \times w + v \times w$ | distributivity |
| Cross[$\{1,2,3\},\{3,4,5\}$] | Mathematica |
| $\frac{d}{dt}(v \times w) = \dot{v} \times w + v \times \dot{w}$ | product rule |

PROJECTIONS.

**Projection:**

$\text{proj}_v(w) = (v \cdot w)v/||v||^2$.

Is a vector parallel to $w$.

**Scalar projection:**

$\text{comp}_v(w) = ||\text{proj}_v(w) = (v \cdot w)/||v||$

Is a number, the length of the projected vector.

SURFACES.

$\{g(x,y,z) = C\}$ define in general surfaces. Examples are **graphs**, where $g(x,y,z) = z - f(x,y) = 0$ or planes, where $g(x,y,z) = ax + by + cz = C$. If $g$ has quadratic or linear terms only, the surface is called a **quadric**: example $x^2 + xy + y^2 = -z^2 + 2x = 0$. Some surfaces are sometimes easier to describe in cylindrical or spherical coordinates: example sphere: $\rho = const$ or cylinder: $r = const$.

Surfaces can be analyzed by looking at **traces**, intersections with planes parallel to the coordinate planes. This is especially true for graphs, where the traces $f(x,y) = C$ are called contour lines. Examples are isobars, isotherms or topolographical contour lines.

QUADRICS CHECKLIST. Quadrics like:

- ellipsoid sphere
- cylinder
- hyperbolic cylinder
- cone
- one sheeted hyperboloid
- two sheeted hyperboloid
- paraboloid
- hyperbolic paraboloid

can be identified using **traces**.

CURVES.

$r(t) = (x(t), y(t), z(t))$, $t \in [a,b]$ defines a curve. By differentiation, we obtain **velocity** $r'(t)$ and **acceleration** $r''(t)$ which are both vectors. If we integrate the speed $||r'(t)||$ over the interval $a, b$, we obtain the **length** of the curve.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Example: $r(t) = (1, 3t^2, t^3)$, $r'(t) = (0, 6t, 3t^2)$, so that $||r'(t)|| = 3t(4+t^2)$. The length of the curve between 0 and 1 is $\int_0^1 3t(4+t^2)\, dt = 6t^2 + 3\frac{t^4}{4}|_0^1 = 6 \cdot \frac{3}{4}$.

COORDINATE SYSTEMS.

| rectangular | cylindrical | spherical |
|---|---|---|
| $(x,y,z)$ | $(r,\theta,z)$ | $(\rho,\theta,\phi)$ |
| $x$ real | $r \geq 0$ | $\rho \geq 0$ |
| $y$ real | $\theta \in [0,2\pi)$ | $\theta \in [0,2\pi)$ |
| $z$ real | $z$ real | $\phi \in [0,\pi]$ |

$x = r\cos(\theta)$
$y = r\sin(\theta)$
$z = z$

$x = \rho\cos(\theta)\sin(\phi)$
$y = \rho\sin(\theta)\sin(\phi)$
$z = \rho\cos(\phi)$