Homework for Thursday: 11.1 24, 31-36 11.2 14,28,34

LEVEL CURVES

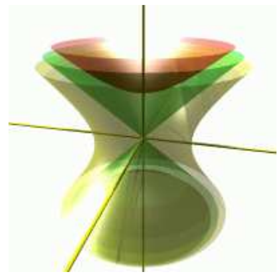
2D: If $f(x,y)$ is a function of two variables, then $f(x,y) = const$ is **curve** in the plane. It is called **contour line** or **level curve**. For example, $f(x,y) = 4x^2 + 3y^2 = 1$ is an ellipse. Level curves allow to visualize the function f.

LEVEL SURFACES.

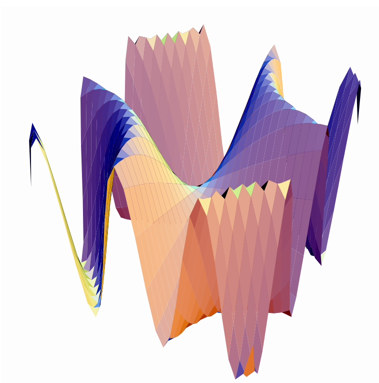
3D: If $f(x,y,z)$ is a function of three variables and c is a constant then $f(x,y,z) = c$ is a surface in space. It is called a **contour surface** or a **level survae**. For example if $f(x,y,z) = 4x^2 + 3y^2 + z^2$, then the contour surfaces are ellipsoids.

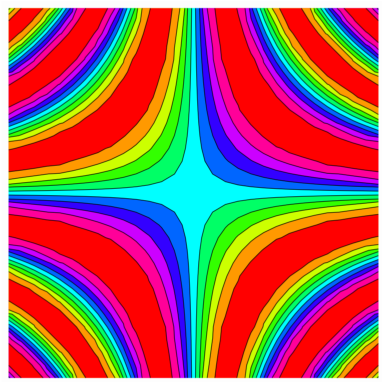
EXAMPLE. Let $f(x,y) = x^2 - y^2$. The set $x^2 - y^2 = 0$ is the union of the sets $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their tips at $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their tips at $(0,\pm 1)$.

EXAMPLE. Let $f(x,y,z) = x^2 + y^2 - z^2$. $f(x,y,z) = 0, f(x,y,z) = 1, f(x,y,z) = -1$. The set $x^2+y^2-z^2 = 0$ is a **cone** rotational symmetric around the z-axes. The set $x^2+y^2-z^2 = 1$ is a **one-sheeted hyperboloid**, the set $x^2 + y^2 - z^2 = -1$ is a **two-sheeted hyperboloid**. (How to see that it is two-sheeted: the intersection with $z = c$ is empty for $-1 \leq z \leq 1$.)

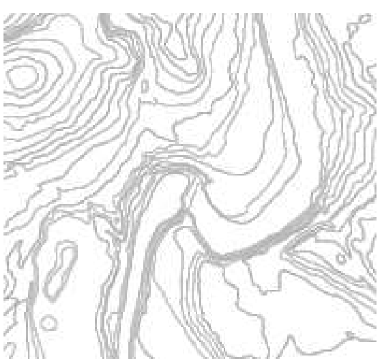


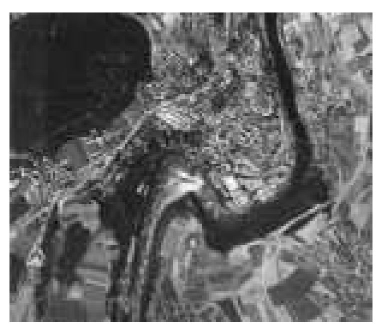
CONTOUR MAP. Drawing several contour lines or surfaces produces a **contour map**.





TOPOGRAPHY. Topographical maps often show the curves of equal height. With this information, it is usually already possible to have a good picture of the situation.

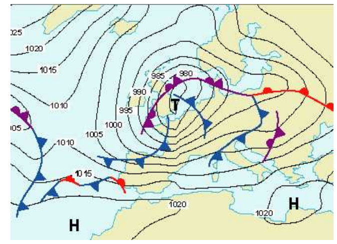




Isobars:	pressure
Isoclines:	direction

Isothermes:	temperature
Isoheight:	height

For example, the isobars to the right show the lines of constant pressure today in Europe.

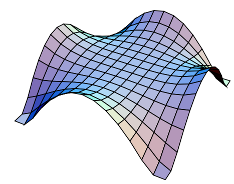


PARTIAL DERIVATIVE. If $f(x,y,z)$ is a function of three variables, then $\frac{\partial}{\partial x} f(x,y,z)$ is defined as the derivative of the function $g(x) = f(x,y,z)$, where y and z are fixed. The other derivatives with respect to y and z are defined similarly.

REMARK. The partial derivatives measure the rate of change of the function in the x, y, z directions.

NOTATION. One also writes $f_x(x,y,z) = \frac{\partial}{\partial x} f(x,y,z)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

EXAMPLE. $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2, f_{xx} = 12x^2 - 12y^2, f_y(x,y) = -12x^2y + 4y^3, f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation itself is called a **partial differential equation** (see separate handout).



CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. Proof. Compare the two sides:

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$
$dy dx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$

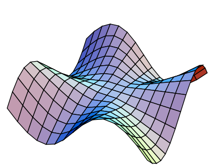
$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$.
$dx dy f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

CONTINUITY IS NECESSARY. Example: $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$ contradicts Clairot:

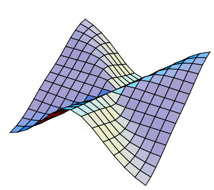
$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x2$, $f_{y,x}(0,0) = 1$.

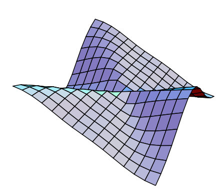
$f(x,y)$



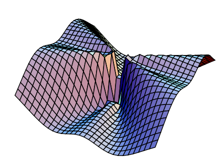
$f_x(x,y)$



$f_y(x,y)$



$f_{xy}(x,y)$



ABOUT CONTINUITY. In reality, one sometimes has to deal with functions which are not smooth: For example, when plotting the temperature of water in relation to pressure and volume, one experiences **phase transitions**, an other example are water waves breaking in the ocean. Mathematicians have also tried to explain "catastrophic" events mathematically with a theory called "catastrophe theory". Discontinuous things are useful (for example in switches), or not so useful (for example, if something breaks).