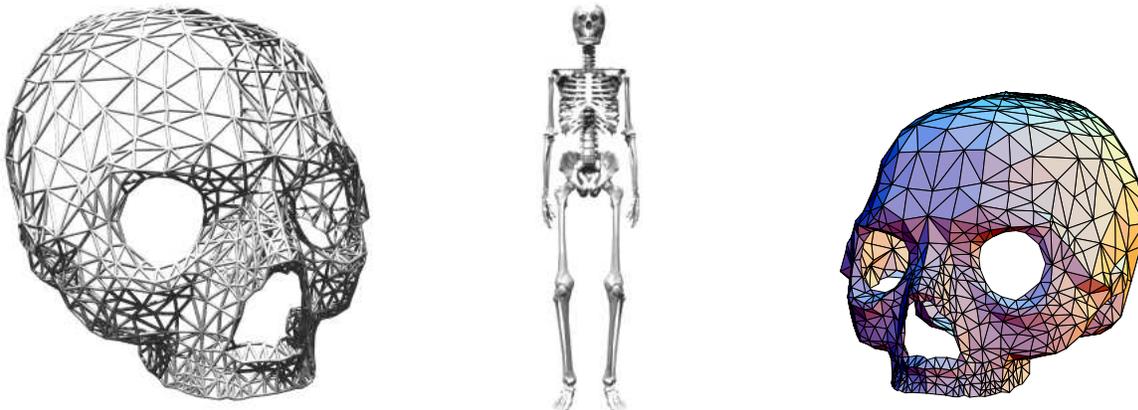


Homework:

11.5 10, 32, 43

11.6 8, 16, 20, 26, 30, 42, 44

HAPPY HALLOWEEN! Here are some Halloween surfaces:

**DIRECTIONAL DERIVATIVE.**

If f is a function of several variables and v is a vector, then $\nabla f \cdot v$ is called the **directional derivative** of f in the direction v . We write $\nabla_v f$.

$$\nabla_v f(x, y, z) = \nabla f(x, y, z) \cdot v$$

It is usually assumed that v is a unit vector. Using the chain rule, one can write $\frac{d}{dt} D_v f = f(x + tv)$.

EXAMPLE. PARTIAL DERIVATIVES ARE SPECIAL DIRECTIONAL DERIVATIVES.

If $v = (1, 0, 0)$, then $\nabla f \cdot v = f_x$.

If $v = (0, 1, 0)$, then $\nabla f \cdot v = f_y$.

If $v = (0, 0, 1)$, then $\nabla f \cdot v = f_z$.

The directional derivative is a generalization of the partial derivatives. Like the partial derivatives, it is a **scalar**.

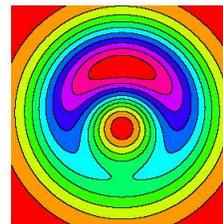
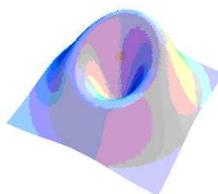
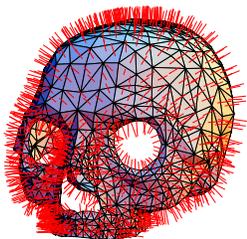
EXAMPLE. DIRECTIONAL DERIVATIVE ALONG A CURVE.

If f is the temperature in a room and $r(t)$ is a curve with velocity $r'(t)$, then $\nabla f(r(t)) \cdot r'(t)$ is the temperature change, the body moving on a curve $r(t)$ experiences: the chain rule told us that this is $d/dt f(r(t))$.

GRADIENTS AND LEVEL SURFACES.

Gradients are orthogonal to the level surfaces.

The level surface of the linear approximation is tangent to the surface. Every vector $\vec{x} - \vec{x}_0$ in this surface $\nabla f \cdot (\vec{x} - \vec{x}_0) = 0$ is orthogonal to ∇f .



STEEPEST DESCENT. The directional derivative satisfies

$$|D_v f| \leq |\nabla f| |v|$$

because $\nabla f \cdot v = |\nabla f| |v| \cos(\phi) \leq |\nabla f| |v|$. The direction $v = \nabla f$ is the direction, where f increases most, the direction $-\nabla f$ is the direction of steepest descent.

IN WHICH DIRECTION DOES f INCREASE? If $v \cdot \nabla f$, then the directional derivative is $\nabla f \cdot \nabla f = |\nabla f|^2$. This means that f **increases**, if we move into the direction of the gradient!

EXAMPLE. You are on a trip in a zeppelin at $(1, 2)$ and want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?



Parameterize the direction by $v = (\cos(\phi), \sin(\phi))$. The pressure gradient is $\nabla p(x, y) = (2x, 4y)$. The directional derivative in the ϕ -direction is $\nabla p(x, y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

ZERO DIRECTIONAL DERIVATIVE. The rate of change in every direction is zero if and only if $\nabla f(x, y) = 0$ because if ∇f is not 0, we can choose $v = \nabla f$ and get $D_{\nabla f} f = |\nabla f|^2$.

We will see later that points with $\nabla f = 0$ are candidates for local maxima or minima of f . Points (x, y) where $\nabla f(x, y) = (0, 0)$ are called **stationary points** or **critical points**. Knowing the critical points is important to understand the function f .

PROPERTIES OF THE DIRECTIONAL DERIVATIVE. The directional derivative D_v has all the properties of a derivative:

$$\begin{aligned} D_v(f + g) &= D_v(f) + D_v(g) \\ D_v(fg) &= D_v(f)g + fD_v(g) \end{aligned}$$

THE MATTERHORN is a popular climbing mountain in the Swiss alps. Its height is 4478 meters (14,869 feet). It is quite easy to climb with a guide. There are ropes and ladders at difficult places. Even so, about 3 people die each year from climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the north-east direction $v = (1, -1)$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = (-2x, -2y)$, so that $(20, -20) \cdot (1, -1) = 40$. This is a place, with a ladder, where you climb 40 meters up when advancing 1m forward.



THE VAN DER WAALS (1837-1923) equation for real gases is

$$(p + a/V^2)(V - b) = RT(p, V),$$

where $R = 8.314 \text{ J/Kmol}$ is a constant called the **Avogadro number**. This law relates the pressure p , the volume V and the temperature T of a gas. The constant a is related to the molecular interactions, the constant b to the finite rest volume of the molecules.



The **ideal gas law** $pV = nRT$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(p, V) = \text{const}$ tell much about the properties of the gas. The so called **reduced van der Waals law** $T(p, V) = (p + 3/V^2)(3V - 1)/8$ is obtained by scaling p, T, V depending on the gas. Calculate the directional derivative of $T(p, V)$ at the point $(p, V) = (1, 1)$ into the direction $v = (1, 2)$. We have $T_p(p, V) = (3V - 1)/8$ and $T_V(p, V) = 3p/8 - (9/8)1/V^2 - 3/(4V^3)$. Therefore, $\nabla T(1, 1) = (1/4, 0)$ and $D_v T(1, 1) = 1/5$.

