11/5/2002, CRITICAL POINTS — Math 21a, O. Knill

CRITICAL POINTS. A point (x_0, y_0) is called a **critical point** of $F(x,y)$ if $\nabla F(x_0,y_0) = (0,0)$. A point (x_0, y_0, z_0) is called a **critical point** of $F(x,y,z)$ if $\nabla F(x_0,y_0,z_0) = (0,0,0)$.

EXAMPLE 1. $F(x,y) = \sin(x^2+y)+y$. The gradient is $\nabla F(x,y) = (2x\cos(x^2+y), \cos(x^2+y)+1)$. Where are the critical points? We must have $x=0$ and $\cos(y)+1=0$ which means $\pi + k2\pi$. The critical points are at $(0,\pi), (0,3\pi)$.

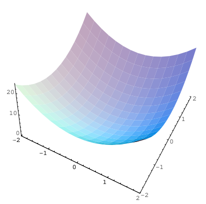
EXAMPLE 2. ("volcano") $F(x,y) = (x^2+y^2)e^{-x^2-y^2}$. The gradient $\nabla F = (2x - 2(x^2+y^2), 2y - 2(x^2+y^2))e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2+y^2=1$. There are ∞ many critical points.

EXAMPLE 3 ("pendulum") $F(x,y) = -g\cos(x) + y^2/2$ is the energy of the pendulum. The gradient $\nabla F = (y, -g\sin(x))$ vanishes for $x = 0, \pi, 2\pi, ..., y = 0$. These points are equilibrium points, where the pendulum is at rest.

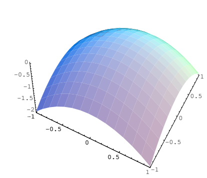
EXAMPLE 4 ("Volterra Lodka") $F(x,y) = a\log(y) - by + c\log(x) - dx$. The maximum occurs at $c/d, a/b$.

EXAMPLE 5 ("From a Advanced Multivariable Calculus Midterm somewhere in the US") Find all the critical points of $f(x,y) = 2x^2 - x^3 - y^2$.

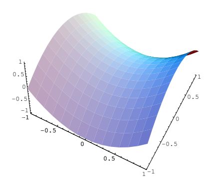
TYPICAL EXAMPLES.



$F(x,y) = x^2 + y^2$

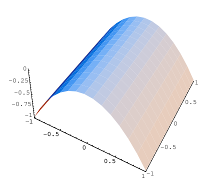


$F(x,y) = -x^2 - y^2$

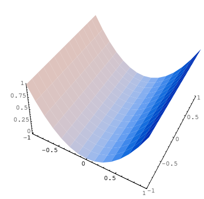


$F(x,y) = x^2 - y^2$

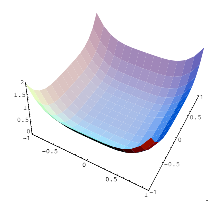
EXAMPLES WITH $\det(H) = 0$.



$F(x,y) = x^2$

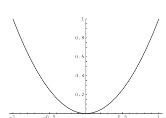


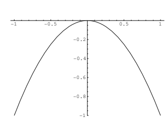
$F(x,y) = -x^2$

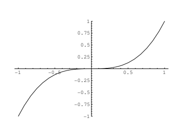


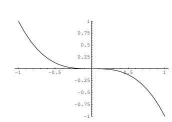
$F(x,y) = x^4 + y^4$

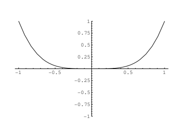
CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSION.
$f'(x) = 0, f''(x) > 0$, local minimum, $f''(x) < 0$ local maximum, $f'' = 0$ undetermined.











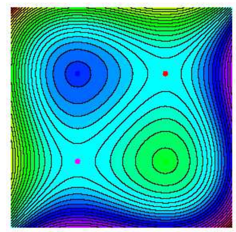
CLASSIFICATION OF CRITICAL POINTS FOR FUNCTIONS OF TWO VARIABLES.
Let $F(x,y)$ be a function of two variables with a critical point (x_0,y_0). Let $H = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix}$ be the **Hessian** at (x_0, y_0).

$\det(H) > 0$ $H_{11} > 0$ local minimum (valley)
$\det(H) > 0$ $H_{11} < 0$ local maximum (top).
$\det(H) < 0$ saddle point (pass).

$\det(H) = 0$, $\mathrm{tr}(H) < 0$ mountain line
$\det(H) = 0$, $\mathrm{tr}(H) > 0$ long valley.
$\det(H) = 0$, $\mathrm{tr}(H) = 0$ need higher derivatives.

EXAMPLE. (A "napkin").
The function $F(x,y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla F(x,y) = (x^2-1, -y^2+1)$. It vanishes at the 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix is $H = F''(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

$F''(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$F''(-1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$F''(1,-1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$F''(-1,-1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$\det(H) = -4$ Saddle point	$\det(H) = 4, H_{11} = -2$ Local maximum	$\det(H) = 4, H_{11} = 2$ Local minimum	$\det(H) = -4$ Saddle point

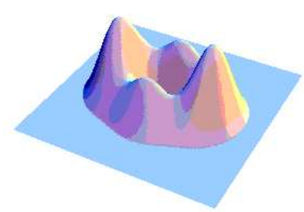


MAXIMA-MINIMA. To determine the maximum or minimum of $f(x,y)$ on a domain domain, determine all critical points **inside the domain**, and compare their values with maxima or minima **at the boundary**.

EXAMPLE 5 Solution. Find all the critical points of $f(x,y) = 2x^2 - x^3 - y^2$. We have $\nabla f(x,y) = 4x - 3x^2, -2y)$. The critical points are at $(4/3,0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{pmatrix} 4-6x & 0 \\ 0 & -2 \end{pmatrix}$. At $(0,0)$, the Hessian determinant is -8 so that this is a saddle point. At $(4/3,0)$, the Hessian determinant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum.

WHY DO WE CARE ABOUT CRITICAL POINTS?
- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature determine a lot about the function.
- Critical points are physically relevant (i.e. configurations with lowest energy).

A CURIOUS OBSERVATION: (The island theorem) Let $F(x,y)$ be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ "charge" if it is a maximum or minium, and with -1 "charge" if it is a saddle point. Sum up all the charges and you will get 1, independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



CRITICAL POINTS IN PHYSICS. (informal) Most physical laws are based on a simple principle: **the equations are critical points of a functional (in general in infinite dimensions). Examples:**
- **Newton equations**. (Classical mechanics) A particle of mass m moving in a field V along a path $\gamma : t \mapsto r(t)$ extremizes the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\ dt$. Critical points γ satisfy the Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$.
- **Maxwell equations**. (Electromagnetism) The electomagnetic field (E,B) extremizes the Integral $S(E,B) = \frac{1}{8\pi}\int (E^2 - B^2)\ dV$ over space time. Critical points are described by the the Maxwell equations in vaccuum.
- **Einstein equations** (General relativity) If g is a dot product which depends on space and time, and R is the "curvature" of the corresponding curved space time, then $S(g) = \int R dV(g)$ is a function of g for which critical points g satisfy the Einstein equations in general relativity.

OTHER WAYS TO FIND CRITICAL POINTS. Some ideas: walk in the direction of the gradient until reaching a local maximum or walkd backwards to reach local minima. To find saddle points, consider the shortest path connecting two local minima and take the maximum along this path.