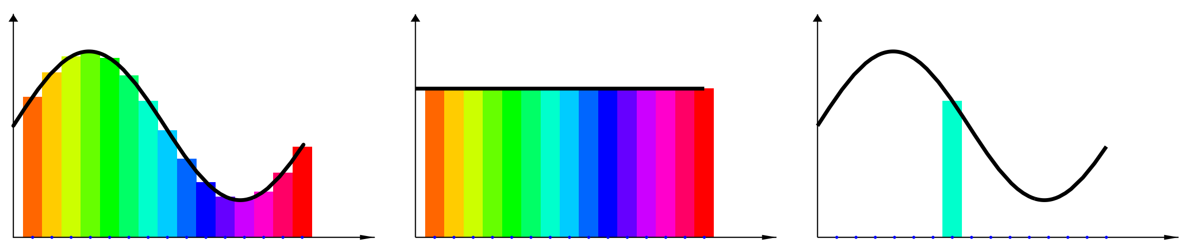
Homework: Section 12.2: 2, 14, 20 12.3 30, 40

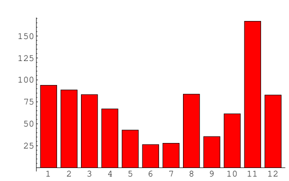
1D INTEGRATION IN 100 WORDS. If $f(x)$ is a continuous function of one variable, then $\int_0^1 f(x)\ dx$ can be defined as a limit of the **Riemann sum** $f_n(x) = \frac{1}{n}\sum_{k=1}^n f(x_k)$ for $n \to \infty$ with $x_k = k/n$. The integral is the **average** of f on the interval $[a,b]$. It can be interpreted as an **signed area** under the graph of f. If $f(x) = 1$, the integral is the **length** of the interval. The function $F(x) = \int_a^x f(y)\ dy$ is called an **anti-derivative** of f. The fundamental theorem of calculus states $F'(x) = f(x)$. Unlike the derivative, anti-derivatives can not always be expressed in terms of known functions: Example: $F(x) = \int_0^x e^{-x^2}\ dx$. Often, the anti-derivative can be found: Example: $f(x) = \sin^2(x) = (\cos(2x)+1)/2, F(x) = x/2 - \sin(2x)/4$.



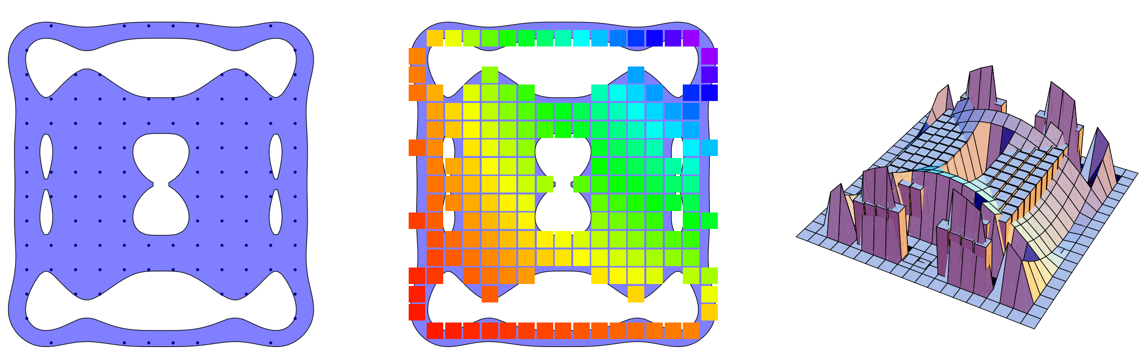
AVERAGES=MEAN. www.worldclimate.com gives the following data for the average monthly rainfall (in mm) for Cambridge, MA, USA (42.38 North 71.11 West,18m Height).

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
93.9	88.6	83.3	67.0	42.9	26.4	27.9	83.8	35.5	61.4	166.8	82.8

The average $860.3/12 = 71.7$ is a Rieman sum integral.



2D INTEGRATION. If $f(x,y)$ is a continuous function of two variables on a region R, the integral $\int_R f(x,y)\ dxdy$ can be defined as the limit $\frac{1}{n^2}\sum_{i,j,x_{i,j}\in R} f(x_i,y_j)$ with $x_{i,j} = (i/n, j/n)$ when n goes to infinity. This integral divided by the area is the **average** value of f on the region R. If $f(x,y) = 1$, then the integral is the **area** of the region R. For many regions, the integral can be calculated as a **double integral** $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\ dy] dx$. In general, the region must be split into pieces, then integrated seperately.



EXAMPLE. Calculate $\int_R f(x,y)\ dxdy$, where $f(x,y) = 4x^2y^3$ and where R is the rectangle $[0,1]\times[0,2]$.

$$\int_0^1 [\int_0^2 4x^2y^3\ dy]\ dx = \int_0^1 [x^2y^4|_0^2]\ dx = \int_0^1 x^2(16-0)\ dx = 16x^3/3|_0^1 = \frac{16}{3}.$$

TYPES OF REGIONS.
$\int\int_R f\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$ **type I region**.
$\int\int_R f\ dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$ **type II region**.
$\int\int_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\ rdrd\theta$ **integral in polar coordinates**.

EXAMPLE. Let R be the triangle $1 \geq x \geq 0, 1 \geq y \geq 0, y \leq x$. Calculate $\int_R e^{-x^2}\ dxdy$.
ATTEMPT. $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$. We can not solve the inner integral because e^{-x^2} has no anti-derivative in terms of elementary functions.
IDEA. Switch order: $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx = \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316....$

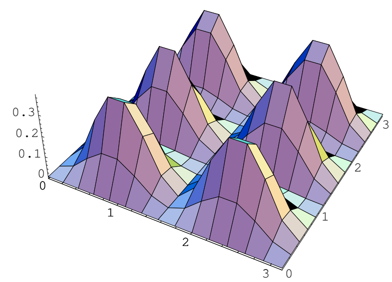
If you can't solve a double integral, try to change the order of integration!

A special case of switching the order of integration is **Fubini's theorem:** $\int_a^b \int_c^d f(x,y)\ \ dxdy = \int_c^d \int_a^b f(y,x)\ dydx$.

QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function $u(x,y)$, the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If R is a region and u is normalized so that $\int |u|^2\ dxdy = 1$, then $\int_R |u(x,y)|^2\ dxdy$ is the **probability**, that the particle is in R.

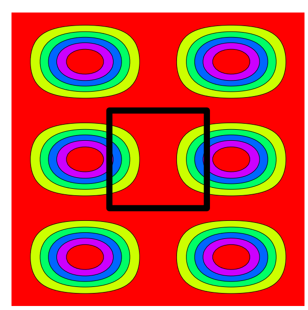
EXAMPLE. Unlike a classical particle, a quantum particle in a box $[0,\pi]\times[0,\pi]$ can have a discrete set of energies only. This is the reason for the name "quantum". If $-(u_{xx}+u_{yy}) = \lambda u$, then a particle of mass m has the energy $E = \lambda\hbar^2/2m$. A function $u(x,y) = \sin(kx)\sin(ny)$ represents a particle of energy $(k^2+n^2)\hbar^2/(2m)$. Our aim is to find the probability that the particle with energy $13\hbar^2/(2m)$ is in the middle 9'th $R = [\pi/3, 2\pi/3]\times[\pi/3, 2\pi/3]$ of the box.
SOLUTION: We first have to normalize $u^2(x,y) = \sin^2(2x)\sin^2(3y)$, so that the average over the whole square is 1:

$$A = \int_0^\pi \int_0^\pi \sin^2(2x)\sin^2(3y)\ dxdy\ .$$



To calculate this integral, we first determine the inner integral $\int_0^\pi \sin^2(2x)\sin^2(3y)\ dx = \sin^2(3y)\int_0^\pi \sin^2(2x)\ dx = \frac{\pi}{2}\sin^2(3y)$ (the factor $\sin^2(3y)$ is treated as a constant). Now, $A = \int_0^\pi (\pi/2)\sin^2(3y)\ dy = \frac{\pi^2}{4}$, so that the **probability amplitude function** is $f(x,y) = \frac{4}{\pi^2}\sin^2(2x)\sin^2(3y)$.

The probability that the particle is in R is slightly smaller than 1/9:



$$\begin{aligned}\frac{1}{A}\int_R f(x,y)\ dxdy &= \frac{4}{\pi^2}\int_{\pi/3}^{2\pi/3}\int_{\pi/3}^{2\pi/3} \sin^2(2x)\sin^2(3y)\ dxdy \\ &= \frac{4}{\pi^2}(4x-\sin(4x))/8|_{\pi/3}^{2\pi/3}(6x-\sin(6x))/12|_{\pi/3}^{2\pi/3} \\ &= 1/9 - 1/(4\sqrt{3}\pi)\end{aligned}$$

The probability is slightly smaller than 1/9.

MOMENT OF INERTIA. Compute the kinetic energy of a square iron plate $R = [-1,1]\times[-1,1]$ of density $\rho = 1$ (about 10cm thick) rotating around its center with a $6'000 rpm$ (rounds per minute). The angular velocity speed is $\omega = 2\pi\ 6'000/60 = 100\ 2\pi$. Because $E = \int\int_R (r\omega)^2/2\ dxdy$, where $r = \sqrt{x^2+y^2}$, we have $E = \omega^2 I/2$, where $I = \int\int_R (x^2+y^2)\ dxdy$ is the **moment of inertia**. For the square, $I = 4/3$. Its energy of the plate is $\omega^2 4/6 = 4\pi^2 100^2 4/6 Joule \sim 0.43 KWh$. You can run with this energy a 60 Watt bulb for 7 hours.

WHERE DO DOUBLE INTEGRALS OCCUR?
- areas.
- averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation of random variables. - quantum mechanics: probability of particle.
- moment of inertia $\int\int_R (x^2+y^2)\rho(x,y)dxdy$
- center of mass $(\int\int_R x\rho(x,y)\ dxdy/M, \int\int_R y\rho(x,y)\ dxdy/M)$, with $M = \int\int_R dxdy$.
- 1D integrals (see challenge problems).