11/14/2002, SURFACE AREA, POLAR COORDINATES Math 21a, O. Knill

Homework: Section 12.4: Numbers 14, 22, 30 section 12.6 Numbers 8, 22

POLAR COORDINATES. For many regions, it is better to use Polar coordinates for integration:

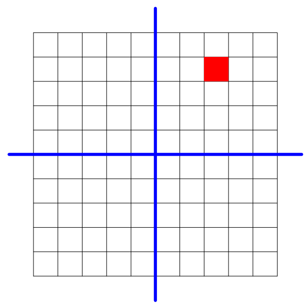
$$\int\int_R f(x,y)\ dxdy = \int\int_R f(r,\theta) r\ drd\theta$$

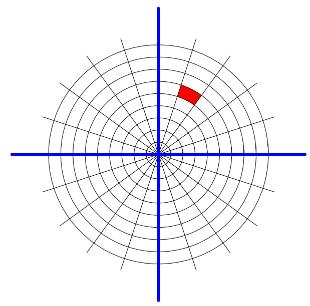
EXAMPLE. The area of the disc $\{x^2+y^2 \leq 1\}$ can be computed by treating the region as a type I region and doing the integral with $x = \sin(u), dx = \cos(u) du$: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\ dydx = \int_{-1}^1 2\sqrt{1-x^2}\ dx = \int_{-\pi/2}^{\pi/2} 2\cos^2(u)\ du = \pi$. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r\ drd\theta = 2\pi r^2/2|_0^1 = \pi\ .$$

WHERE DOES THE FACTOR "r" COME FROM?

A small rectangle with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment in the (x,y) plane. It has approximately the area $rd\theta dr$. It is small for small r.



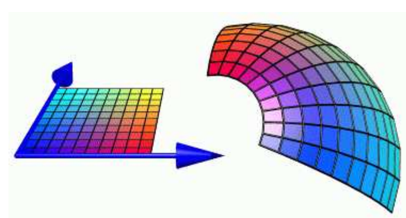


SURFACE AREA

$$\int\int_R |X_u(u,v) \times X_v(u,v)|\ dudv$$

is the area of the surface.

INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If S is a surface, then $\int\int_S f(x,y)\ dS$ should be an average of f on the surface. If $f(x,y) = 1$, then $\int\int_S\ dS$ should be the area of the surface. If S is the image of X under the map $(u,v) \mapsto X(u,v)$, then $dS = |X_u \times X_v| dudv$.



DEFINITION. Given a surface $S = X(R)$, where R is a domain in the plane and where $X(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface integral of $f(u,v)$ on S is defined as

$$\int\int_S f\ dS = \int\int_R f(u,v)|X_u \times X_v|\ dudv\ .$$

INTERPRETATION. If $f(x,y)$ measures a quantity then $\int\int_S f\ dS$ is the average of the function f on S.

EXPLANATION OF $|X_u \times X_v|$. The vector X_u is a tangent vector to the curve $u \mapsto X(u,v)$, when v is fixed and the vector X_v is a tangent vector to the curve $v \mapsto X(u,v)$, when u is fixed. The two vectors span a parallelogram with area $|X_u \times X_v|$. A little rectangle spanned by $[u, u+du]$ and $[v, v+dv]$ is mapped by X to a parallelogram spanned by $[X, X+X_u]$ and $[X, X+X_v]$.

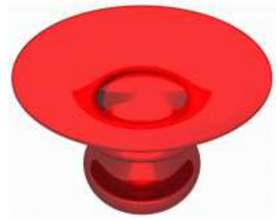
A simple case: consider $X(u,v) = (2u, 3v, 0)$. This surface is part of the x-y plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $X_u \times X_v = (0,0,6)$ and we see for example that the area of $X(R)$ is 6 times the area of R.

THE AREA OF THE SPHERE. The map $X = (u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ maps the rectangle $R : [0,2\pi] \times [0,\pi]$ onto the sphere. $X_u \times X_v = \sin(v)X(u,v)$. So, $|X_u \times X_v| = |\sin(v)|$ and $\int\int_R 1 dS = \int_0^{2\pi} \int_0^{\pi} \sin(v)\, dvdu = 4\pi$.

AREA OF GRAPHS. For surfaces $(u,v) \mapsto (u,v,f(u,v)$, we have $X_u = (1,0,f_u(u,v))$ and $X_v = (0,1,f_v(u,v))$. The cross product $X_u \times X_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region R is $\int\int_R \sqrt{1+f_u^2+f_v^2}\, dA$.

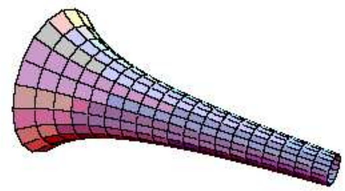
EXAMPLE. The surface area of the paraboloid $z = f(x,y) = x^2+y^2$ is (use polar coordinates) $\int_0^{2\pi} \sqrt{1+4r^2} r\, drd\theta = 2\pi(2/3)(1+4r^2)^{3/2}/8|_0^1 = \pi(5^{3/2}-1)/6$.

AREA OF SURFACES OF REVOLUTION. If we rotate the graph of a function $f(x)$ on an interval $[a,b]$ around the x-axes, we get a surface parameterized by $(u,v) \mapsto X(u,v) = (v, f(v)\cos(u), f(v)\sin(u))$ on $R = [0,\pi] \times [a,b]$ and is called a **surface of revolution**. We have $X_u = (0, -f(v)\sin(u), f(v)\cos(u)), X_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $X_u \times X_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$ which has the length $|X_u \times X_v| = |f(v)|\sqrt{1+f'(v)^2}\, dudv$.



EXAMPLE. If $f(x) = x$ on $[0,1]$, we get the surface area of a cone: $\int_0^{2\pi} \int_0^1 x\sqrt{1+1}\, dvdu = 2\pi\sqrt{2}/2 = \pi\sqrt{2}$.

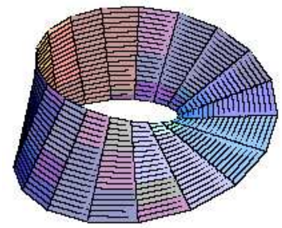
GABRIEL'S TRUMPET. Take $f(x) = 1/x$ on the interval $[1,\infty)$.
Volume: The volume is (use cylindrical coordinates in the x direction): $\int_1^\infty \pi f(x)^2\, dx = \pi \int_1^\infty 1/x^2\, dx = \pi$.
Area: The area is $\int_0^{2\pi} \int_1^\infty 1/x\sqrt{1+1/x^4}\, dx \geq 2\pi \int_1^\infty 1/x\, dx = 2\pi \log(x)|_0^\infty = \infty$.



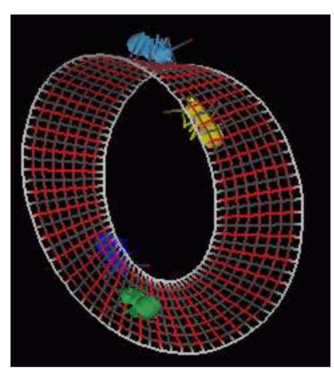
The Gabriel trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

Question. How long does a Gabriel trumpet have to be so that its surface is $500cm^2$ (area of sheet of paper)? Because $1 \leq \sqrt{1+1/x^4} \leq \sqrt{2}$, the area for a trumpet of length L is between $2\pi \int_1^L 1/x\, dx = 2\pi\log(L)$ and $\sqrt{2}2\pi\log(L)$. In our case, L is between $e^{500/(\sqrt{2}2\pi)} \sim 2 * 10^{24} cm$ and $e^{500/(2\pi)} \sim 4 * 10^{34} cm$. Note that the universe is about 10^{26} cm long! It could hardly accomodate a Gabriel trumpet in our universe if it should have the surface area of a sheet of paper!

MÖBIUS STRIP. The surface $X(u,v) = (2 + v\cos(u/2)\cos(u), (2+v\cos(u/2))\sin(u), v\sin(u/2))$ parametrized by $R = [0,2\pi] \times [-1,1]$ is called a **Möbius strip**.
The calculation of $|X_u \times X_v| = 4+3v^2/4+4v\cos(u/2)+v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0,2\pi] \times [-1,1]$ is 17π.



QUESTION. If we build the Moebius strip from paper. What is the relation between the area of the surface and the weight of the surface?



REMARKS.
1) An OpenGL implementation of an Escher theme can be admired with
"xlock -inwindow -mode moebius"
on an X-terminal. 2) A patent was once assigned to the idea to use a Moebius strip as a **conveyor belt**. It would last twice as long as an ordinary one.