11/19/2002, REVIEW (second Midterm) **Math 21a, O. Knill**

MIDTERM TOPICS.

- Partial derivatives, gradient
- Extrema of functions of two variables.
- Extrema of functions with constraints.
- Parameterized surfaces.
- Integration of functions of two variables.
- Chain rule
- Linear approximation.
- Estimation using linear approximation.
- Tangent planes.
- Integration in Polar coordinates

LEVELS OF UNDERSTANDING.

1. I) KNOW. Know **what** the objects, definitions, theorems, names are. Know jargon and history. (You learn this mostly in class or by reading the book).
2. II) DO. Know **how** to work with the objects. Persue algorithms. (You learn this mostly by doing homework, doing in class exercices).
3. III) UNDERSTAND. See different aspects, contexts. **Why** is it done like this? (You learn this mostly from listening to lectures, discussions and by gaining experience).
4. IV) APPLY. Extend the theory, apply to new situations, invent new objects. Ask: **Why not...?** (You learn this mostly by doing challenging problems, experiment with technology, writing papers).

I) Definitions and objects.

CONSTRAINED EXTREMUM of f constrained by $G = c$ are obtained where $\nabla f = \lambda \nabla g, g = c$.
CHAIN RULE. If $r(t) = (x(t), y(t))$ is a curve and $f(x, y)$ is a function, then $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.
CHAIN RULE. If $g(x)$ and $f(x, y)$ are functions, then $(\partial/\partial x)g(f(x, y)) = g'(f(x, y))f_x(x, y)$, $(\partial/\partial y)g(f(x, y)) = g'(f(x, y))f_y(x, y)$.
CRITICAL POINT. $\nabla f(x, y) = (0, 0)$. Is also called **stationary point**.
DOUBLE INTEGRAL. $\int_a^b \int_{f(x)}^{g(x)} f(x, y)\, dydx$ is an example of a double integral.
2D POLAR INTEGRAL. $\int\int_R f(r, \phi)\, r dr d\phi$ in polar coordinates.
GRADIENT. $f(x, y)$ function of two variables, $\nabla f(x, y) = (\partial_x f(x, y), \partial_y f(x, y)) = (f_x(x, y), f_y(x, y))$.
HESSIAN MATRIX $f(x, y)$ function of two variables. The Hessian is the matrix $H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$.
HESSIAN DETERMINANT. $D = f_{xx}(x, y)f_{yy}(x, y) - fxy^2(x, y)$. LEVEL SURFACE. $f(x, y, z) = 0$ has gradients $\nabla f(x, y, z)$ as normals.
LINEAR APPROXIMATION. $L(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$.
LOCAL MAXIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) < 0$ is a local maximum.
LOCAL MINIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) > 0$ is a local minimum.
PARTIAL DIFFERENTIAL EQUATION. Equation for a function. Involves partial derivatives of the function. Example: $f_{tt} = f_{xx}$ wave equation, $f_t = f_{xx}$ heat equation.
SADDLE POINT. A critical point for which $\det(H(x, y)) < 0$.
SECOND DERIVATIVE TEST. $D < 0 \Rightarrow$ saddle, $D > 0, f_{xx} > 0 \Rightarrow$ min, $D > 0, f_{yy} < 0 \Rightarrow$ max.

II) Algorithms

INTEGRATION OVER A DOMAIN R.
1) Eventually chop the region into pieces which can be parametrized.
2) Start with one variable, say x and find the smallest x-interval $[a, b]$ which contains R.
3) For fixed x, intersect the line $x = const$ with R to determine the y-bounds $[f(x), g(x)]$.
4) Evaluate the integral $\int_a^b \left[\int_{f(x)}^{g(x)} f(x, y)\right] dydx$.
5) Solve the double integral by 1D integration starting from inside.
6) In case of problems with the integral, try to switch the order of integration. (Go to 2) and start with y).

EXAMPLE. Integrate x^2y^2 over the triangle $x + y/2 \leq 3, x > 0, y > 1$. The triangle is contained in the strip $0 \leq x \leq 3$. The x-integration ranges over the interval $[0, 3]$. For fixed x, we have $y \geq 1$ and $y \leq 2(3 - x)$ which means that the y bounds are $[0, 2(3 - x)]$. The double integral is $\int_0^3 \int_1^{6-2x} x^2y^2\, dydx$.

FINDING THE MAXIMUM OF A FUNCTION $f(x,y)$ OVER A DOMAIN R with boundary $g(x,y) = c$.

1) First look for all stationary points $\nabla f(x,y)$ in the interior of R.
2) Eventually classify the points in the interior by looking at $\det(H)(x,y)$, $H_{xx}(x,y)$ at the critical points.
3) Locate the critical points at the boundary by solving $\nabla F(x,y) = \lambda \nabla G(x,y), G(x,y) = c$.
4) List the values of F evaluated at all the points found in 1) and 3) and compare them.

EXAMPLE. Find the maximum of $F(x,y) = x^2 - y^2 - x^4 - y^4$ on the domain $x^4 + y^4 \leq 1$. $\nabla F(x,y) = (2x - 4x^3, -2y - 4y^3)$. The critical points inside the domain are obtained by solving $2x - 4x^3 = 0, -2y - 4y^3 = 0$ which means $x = 0, x = 1/4$, $y = 0, y = -1/4$. We have four points $P_1 = (0,0), P_2 = (1/\sqrt{2}, 0), P_3 = (0, -1/\sqrt{2}), P_4 = (1/\sqrt{2}, -1/\sqrt{2})$.
The critical points on the boundary are obtained by solving the Lagrange equations $(2x - 4x^3, -2y - 4y^3) = \lambda(4x^3, 4y^3), x^4 + y^4 = 1$. Solutions (see below) are $P_5 = (0,-1), P_6 = (0,1), P_7 = (-1,0), P_8 = (1,0)$. A list of function values $F(P_1) = 0, F(P_2) = 1/2 - 1/4, F(P_3) = -1/2 - 1/4, F(P_4) = -2/4, F(P_5) = -2, F(P_6) = -2, F(P_7) = 0, F(P_8) = 0$ shows that P_2 in the interior is the maximum. Indeed, the Hessian at this point is $H = \text{diag}(-1, -2)$ which has positive determinant and negative H_{11}.

SOLVING THE LAGRANGE EQUATIONS.
1) Write down the equations neatly.
2) See whether some variable can be eliminated easily. λ can always be eliminated.
3) If some variable can be eliminated easily, go back to 1) using one variable less and repeat.
4) Try to combine, rearrange, simplify the equations. The system might not have an algebraic solution.

EXAMPLE.

$$\begin{aligned} 2x - 4x^3 &= 4\lambda x^3 \\ -2y - 4y^3 &= 4\lambda y^3 \\ x^4 + y^4 &= 1 \end{aligned}$$

$$\begin{aligned} 2x - (4 + 4\lambda)x^3 &= 0 \\ -2y - (4 + 4\lambda)y^3 &= 0 \\ x^4 + y^4 &= 1 \end{aligned}$$

$$\begin{aligned} x = 0 \quad &\text{or} \quad x = 1/\sqrt{2 + 2\lambda} \\ y = 0 \quad &\text{or} \quad y = -1/\sqrt{2 + 2\lambda} \\ x^4 + y^4 &= 1 \end{aligned}$$

If x=0, then y=1, or y=-1, if $y = 0$ then $x = 1$ or $x = -1$. There are 4 critical points.

III) "Understanding". Try to answer questions like:

- What do the Lagrange equations mean geometrically? Explain the method to somebody with a drawing.
- Explain why only critical points can be candidates for maxima or minima of a function $F(x,y)$.
- What can happen at a critical point if the discriminant $f_{xx}f_{yy} - f_{xy}^2$ is 0 at this point?
- What is the geometric meaning of the entry H_{xy} in the Hessian?
- Discuss the chain rule $F(r(t))$ for $F(x,y) = \sqrt{x^2}$, $r(t) = (x(t), y(t)) = (t, t^2)$.
- In which situations is the Lagrange multiplyer $\lambda = 0$?
- If a function $f(x,y)$ is replaced by its linear approximation $L(x,y)$, what do you expect the error $L(0.01, 0.01) - f(0.01, 0.01)$ to be?
- How can the chain rule be used to find $z_x(x,y)$ if $f(x, y, z(x,y)) = 0$?
- Can we have functions which contain two saddles as critial points and no other critical points?
- Find the second partial derivatives of $g(f(x,y))$.

IV) Apply to new situations.

Problem solving and creativity skills are acquired best by "doing" it, by pondering over new questions, working on specific problems. Nevertheless, there is also theoretical help: for example, G. Polya (1887-1985)'s book "how to solve it" gives some general advise on "how to solve problems". Abstract: "How to solve a problem?":

1) Understand the problem.
2) Think of a plan by solving subproblems. Connect with older problems.
3) Walk along the plan while controlling each step.
4) Check the result. Is the result obvious? Is the method useful for other problems?