

REMINDER: INTEGRATION POLAR COORDINATES.

$$\int_0^r \int_0^{2\pi} f(r, \theta) r d\theta dr .$$

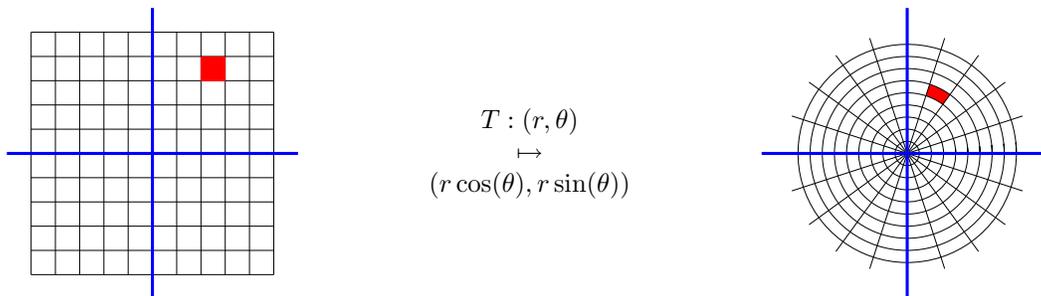
EXAMPLE 1. Area of a disk of radius R

$$\int_0^R \int_0^{2\pi} r d\theta dr = 2\pi \frac{r^2}{2} \Big|_0^R = R^2 \pi .$$

WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r, θ) plane is mapped to a sector segment in the (x, y) plane. It has approximately the area $r d\theta dr$. It is small for small r .

2. EXPLANATION. Look at the map $T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$ which changes from Cartesian coordinates to polar coordinates. The Jacobian is $T' = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ r \sin(\theta) & r \cos(\theta) \end{bmatrix}$ is a matrix with determinant r . This is a special case of a more general formula.



SUBSTITUTION IN HIGHER DIMENSIONS (informal). If $(x, y) = T(u, v)$ is an arbitrary change of variables, the formula

$$\int \int_{T(R)} f(x, y) dx dy = \int \int_R f(u, v) \det(T') du dv$$

holds. The substitution formula is the same as in one dimensions, just take $\det(T')$ instead of T' . The polar coordinates $(x, y) = T(r, \theta) = (r \cos(\theta), r \sin(\theta))$ is an example.

1D CASE: $x = T(u)$, $dx = T'(u)du$ for 1D definite integrals:

$$\int_{T(a)}^{T(b)} f(x) dx = \int_a^b f(T(u))T'(u)du .$$

Ex: $\int_{-1}^1 \sqrt{1-x^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u) du$, $x = T(u) = \sin(u)$, $dx = T'(u)du = \cos(u)du$, $f(T(u)) = \sqrt{1 - \sin^2(u)}$.

CYLINDRICAL COORDINATES. This are polar coordinates in the x-y plane and leaving the z coordinate. Take $T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$. The integration factor $r = \det(T')$ is as in polar coordinates.

$$\int \int \int_{T(R)} f(x, y, z) dx dy dz = \int \int \int_R f(r, \theta, z) r dr d\theta dz$$

COORDINATES OF CAMBRIDGE. On the website <http://cello.cs.wiuc.edu/cgi-bin/slam/ipl2ll/> you can enter a host like www.math.harvard.edu and get latitude and longitude of the host: $(lat, lon) = (42.365, -71.1)$. Using (r, θ, ϕ) coordinates, we obtain the position $(r, 90 - 42.365, -71.1)$ of the host in spherical coordinates. The site does not give the height, but we are about on sea-level, so that $r = 6365km$.

EXAMPLE. Calculate the volume bounded by the parabolic $z = 1 - (x^2 + y^2)$ and the $x - y$ plane. In cylindrical coordinates, the paraboloid is $z(r, \phi) = 1 - r^2$:

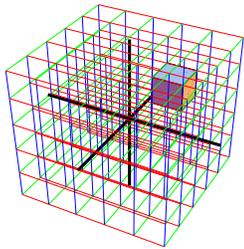
$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r dz d\phi dr = \int_0^1 \int_0^{2\pi} (r - r^3) d\phi dr = 2\pi(r^2/2 - r^4/4)_0^1 = \pi.$$

SPHERICAL COORDINATES. Spherical coordinates use two angles, the longitude θ , the radius ρ as well as ϕ , the angle between the vector (x, y, z) and the z axes. Using these two angles and r , the coordinate change is

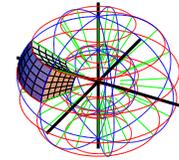
$$T : (x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)).$$

The integration factor can be seen from the dimensions of a spherical wedge with dimensions $d\rho, \rho \sin(\phi)d\theta, \rho d\phi$.

$$\int \int \int_{T(R)} f(x, y, z) dx dy dz = \int \int \int_R f(\rho, \theta, z) \rho^2 \sin(\phi) d\rho d\theta d\phi$$



$$T : (\rho, \theta, \phi) \mapsto (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi))$$



VOLUME OF SPHERE. A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin(\phi) d\phi d\theta d\rho.$$

The most inner integral $\int_0^\pi \rho^2 \sin(\phi) d\phi = -\rho^2 \cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2 d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2 d\rho = 4\pi R^3/3$.

MOMENT OF INERTIA. The moment of inertia of a body G with respect to an axes L is the triple integral $\int \int \int_G r(x, y, z)^2 dz dy dx$, where $r(x, y, z) = R \sin(\phi)$ is the distance from L . Problem: calculate the moment of inertia of a sphere of radius R with respect to the z -axes:

$$I = \int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) d\phi d\theta d\rho = \left(\frac{1}{3} \sin^3(\phi)\right)_0^\pi \left(\int_0^R \rho^4 d\rho\right) \left(\int_0^{2\pi} d\theta\right) = 4/3 \cdot R^5/5 \cdot 2\pi = \frac{8\pi R^5}{15} = \frac{VR^2}{5}.$$

If a sphere spins around the z -axes with angular velocity ω , then $I\omega^2/2$ is the kinetic energy of that sphere. Example: the moment of inertia of the earth is $810^{37}kgm^2$. The angular velocity is $\omega = 1/day = 1/(86400s)$ so that the energy of the earth rotation is $810^{37}kgm^2/(7464960000s^2) \sim 10^{28}J = 10^{25}kJ \sim 2.510^{24}kcal$. How long would you have to run on a treadmill to achieve this energy if you could make 2500 kcal in 1 hour? We would have to run 10^{21} hours = 3.610^{24} sec. The universe is about 10^{17} seconds old. If all the 6 million people in Massachusetts would have run since the big bang on a treadmill, they could have produced the necessary energy to bring the earth to the current rotation ...