HOMEWORK:

PLANAR VECTOR FIELD EXAMPLES.
1) $F(x, y) = (y, -x)$ is a planar vector field which you see in a picture on the right.
2) $F(x, y) = (x - 1, y)/((x - 1)^2 + y^2)^{3/2} - (x + 1, y)/((x + 1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. See picture.
3) **Gradient field**. If $f(x, y, z)$ is a function of three variables, then $\nabla f(x, y, z)$ is called the **gradient field** or **conservative**. For example, $(2x, 2y, -2z)$ is the vector field which is orthogonal to hyperboloids.
4) If $H(x, y)$ is a function of two variables, then $(H_y(x, y), -H_x(x, y))$ is called a **Hamiltonian vector field**. Example: Harmonic Oscillator $H(x, y) = x^2 + y^2$. $(H_y(x, y), H_x(x, y)) = (y, -x)$ is the vector field in 1).
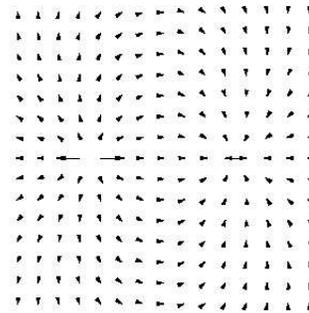
LINE INTEGRALS. If $F(x, y, z)$ is a vector field and $\gamma : t \mapsto r(t)$ is a curve, then $\boxed{\int_a^b F(r(t)) \cdot r'(t)\, dt}$ is called the **line integral** of F along the curve γ. The short-hand notation $\int_\gamma F \cdot ds$ is also used. In the literature, where curves are sometimes written as $r(t) = (x(t), y(t), z(t))$ or $r(t)$, the notation $\int_\gamma F \cdot dr$ or $\int_\gamma F \cdot dr$ appears.

EXAMPLE: **Work**. If $F(x, y, z)$ is a force field, then the line integral $\int_a^b F(r(t)).r'(t)\, dt$ is called **work**.

EXAMPLE: **Electric potential**. If $E(x, y, z)$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r'(t)\, dt$ is called **electric potential**.

EXAMPLE: **Gradient field**. If $F(x, y, z) = \nabla U(x, y, z)$ is a gradient field, then $\int_a^b F(r(t)) \cdot r'(t)\, dt = U(r(b)) - U(r(a))$. The gradient field has physical relevance. For example, if $U(x, y, z)$ is the pressure distribution in the atmosphere, then $\nabla U(x, y, z)$ is the pressure gradient. It coincides roughly with the wind velocity field.

EXAMPLE 1. Let $\gamma : t \mapsto r(t) = (\cos(t), \sin(t))$ be a circle parametrized by $t \in [0, 2\pi]$ and let $F(x, y) = (-y, x)$. Calculate the line integral $I = \int_\gamma F(r) \cdot dr$.

ANSWER: We have $I = \int_0^{2\pi} F(r(t)) \cdot r'(t)\ dt = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t))\ dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t)\ dt = 2\pi$
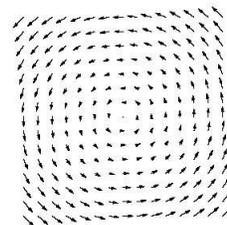
EXAMPLE 2. Let $r(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let F be the vector field $F(x, y) = (-xy, 0)$. Calculate the line integral $\int_\gamma F \cdot dr$.

SOLUTION. In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

$$\int_0^\pi F(r(t)) \cdot r'(t)\ dt = \int_0^\pi (\cos^4(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\ dt$$

$$= -2 \int_0^\pi \sin^2(t)\cos^4(t)\ dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8$$

$$\int_a^b F(r(t)) \cdot r'(t) \, dt = U(r(b)) - U(r(a))$$

The line integral gives the potential difference between the points $r(b)$ and $r(a)$.

EXAMPLE. Let $U(x, y, z)$ be the temperature distribution in a room and let $r(t)$ the path of a fly in the room, then $U(r(t))$ is the temperature, the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt}U(r(t))$. The line-integral of the temperature gradient ∇U along the path of the fly coincides with the temperature difference.

SPECIAL CASES.
$r(t)$ parallel to level curve means $d/dt U(r(t)) = 0$ and $r'(t)$ orthogonal to $\nabla U(r(t))$
$r(t)$ orthogonal to level curve means $|d/dt U(r(t))| = |\nabla U||r'(t)|$ and $r'(t)$ parallel to $\nabla U(r(t))$.

PROOF OF THE FUNDAMENTAL THEOREM. Use the chain rule in the second equality and the fundamental theorem of calculus in the third equality:

$$\int_a^b F(r(t)) \cdot r'(t) \, dt = \int_a^b \nabla U(r(t)) \cdot r'(t) \, dt = \int_a^b \frac{d}{dt}U(r(t)) \, dt = U(r(b)) - U(r(a)) \ .$$

ADDING AND SUBTRACTING CURVES.

If γ_1, γ_2 are curves, then $\gamma_1 + \gamma_2$ denotes the curve obtained by traveling first along γ_1, then along γ_2. One writes $-\gamma$ for the curve γ traveled backwards and $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$.

EXAMPLES. If $\gamma_1(t) = (t, 0)$ for $t \in [0, 1]$, $\gamma_2(t) = (1, (t-1))$ for $t \in [1, 2]$, $\gamma_3(t) = (1 - (t-2), 1)$ for $t \in [2, 3]$, $\gamma_4(t) = (0, 1 - (t-3))$ for $t \in [3, 4]$, then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ for $t \in [0, 4]$ is the path which goes around a the unit square. The path $-\gamma$ travels around in the clockwise direction.
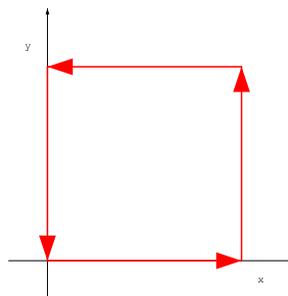

CALCULATING WITH LINE-INTEGRALS.

- $\int_\gamma F \cdot dr + \int_\gamma G \cdot dr = \int_\gamma (F + G) \cdot dr$.

- $\int_\gamma cF \cdot dr = c \int_\gamma F \cdot dr$.

- $\int_{\gamma_1 + \gamma_2} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr$

- $\int_{-\gamma} F \cdot dr = - \int_\gamma F \cdot dr$.

CLOSED CURVES. We see from the last example that the line integral along a closed curve is zero if the vector field is a gradient field. The work done along a closed path is zero. This is a form of **energy conservation**.

PERPETUUM MOBILES. A machine which implements a force field which is not a gradient field is called a **perpetuum mobile**. Mathematically, it realizes a force field for which there exist some closed loops along which the energy gain is nonnegative. (By possibly changing the direction, the energy change can be made positive). The first law of thermodynamics forbids the existence of such a machine. It is informative to stare at some of the ideas people have come up with and to see why they don't work. The drawings of Escher appear also to produce situations, where a force field can be used to gain energy. Escher uses genius graphical tricks however.
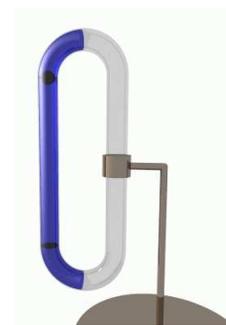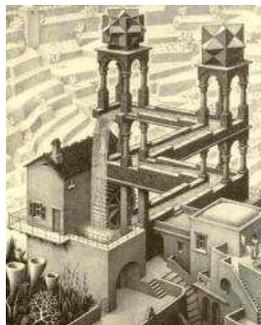
WHEN IS A VECTOR FIELD A GRADIENT FIELD (2D)?

$F(x, y) = \nabla U(x, y)$ implies $F_y(x, y) = F_x(x, y)$. If this does not hold at some point, F is no gradient field. We will see in the next hour, how the condition $\operatorname{curl}(F) = F_y - F_x = 0$ can assure that F is conservative.