Homework: (due 12/10)
13.3 16, 26
13.4 8, 18, 20
13.5 4, 6, 10, 12, 34

LINE INTEGRALS (RECALL). If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field and $\gamma : t \mapsto r(t) = (x(t), y(t)), t \in [a,b]$ is a cuve, then

$$\int_\gamma F\ ds = \int_a^b F(x(t), y(t)) \cdot (x'(t), y'(t))\ dt$$

is called the **lineintegral of F along γ**.

THE CURL OF A 2D VECTOR FIELD. The curl of a 2D vector field $F(x,y) = (P(x,y), Q(x,y))$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) .$$

INTERPRETATION. curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times F = \text{curl}(F)$ for the curl of F because the cross product of (∂_x, ∂_y) with F is the expression.

EXAMPLES.
1) $F(x,y) = (-y, x)$. $\text{curl}(F)(x,y) = 2$.
2) $F(x,y) = \nabla U$, (conservative field = gradient field = potential) Because $P(x,y) = U_x(x,y), Q(x,y) = U_y(x,y)$, we have $Q_x - P_y = U_{yx} - U_{xy} = 0$.

GREEN'S THEOREM. (1827) If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field in the plane and R is a region in the plane which has as a boundary a piecewise smooth closed curve γ traversed in the direction so that the region R is "to the left". Then

$$\int_\gamma F \cdot ds = \int \int_R \text{curl}(F)\ dxdy$$

Note: for a region with holes, the boundary consists of many curves. They are always oriented so that R is to the left.

GEORGE GREEN (1793-1841) was one of the most remarkable of nineteenth century physicists, a self-taught mathematician whose work has contributed greatly to modern physics. Unfortunately, there is no picture of Green.

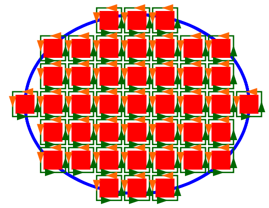
EXAMPLE. If F is a gradient field, then both sides of Green's theorem are zero.

$\int_\gamma F \cdot ds$ vanishes by the fundamental theorem for line integrals.	$\int_R \text{curl}(F) \cdot ds$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(U)) = 0$.

The fact that $\text{curlgrad}(U) = 0$ can also be seen from $\nabla \times \nabla U$ and the general fact that the cross product of two identical vectors is 0. Just treat ∇ as a vector.

WHERE IS THE PROOF? (Quote: General Hein in "Final Fantasy").
To prove Green's theorem look first at a small square $R = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $F = (P, Q)$ along the boundary is $\int_0^\epsilon P(x+t, y)dt + \int_0^\epsilon Q(x+\epsilon, y+t)\, dt - \int_0^\epsilon P(x+t, y+\epsilon)\, dt - \int_0^\epsilon Q(x, y+t)\, dt$. (Note also that this line integral measures the "circulation" at the place (x, y).)

Because $Q(x+\epsilon, y) - Q(x, y) \sim Q_x(x, y)\epsilon$ and $P(x, y+\epsilon) - P(x, y) \sim P_y(x, y)\epsilon$, the line integral is $(Q_x - P_x)\epsilon^2$ is about the same as $\int_0^\epsilon \int_0^\epsilon \mathrm{curl}(F)\, dxdy$. All identities hold in the limit $\epsilon \to 0$.
To prove the statement for a general region R, we chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.



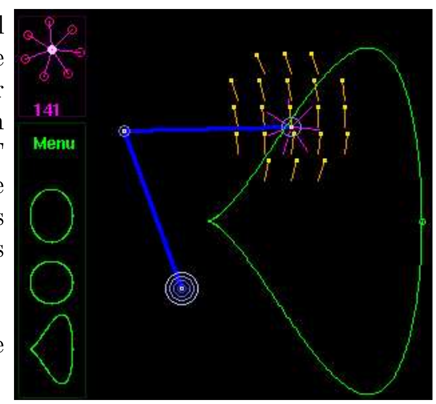
APPLICATION: AREA FORMULAS. The vector fields $F(x, y) = (P, Q) = (-y, 0)$ or $F(x, y) = (0, x)$ have vorticity $\mathrm{curl}(F(x, y)) = 1$. The right hand side in Green's theorem is the **area** of R:

$$Area(R) = \int_\gamma -y\, dx = \int_\gamma x\, dy$$

EXAMPLE. Let R be the region under the graph of a function $f(x)$ on $[a, b]$. The lineintegral around the boundary of R is 0 from $(a, 0)$ to $(b, 0)$ because $F(x, y) = 0$ there. The lineintegral is also zero from $(b, 0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a, 0)$ because $Q = 0$. The line integral on $(t, f(t))$ is $-\int_a^b((-y(t), 0) \cdot (1, f'(t))dt = \int_a^b f(t)\, dt$. Green's theorem assures that this is the area of the region below the graph.

APPLICATION. THE PLANIMETER. The planimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in ingeneering to measure the size of profiles. There is a vector field F associated to a planimeter (put a vector of length 1 orthogonally to the arm). One can prove that F has vorticity 1. The planimeter calculates the line integral of F along a given curve. Green's theorem assures it is the area.

The picture to the right shows a Java applet which allows to explore the planimeter (from a CCP module by O. Knill and D. Winter, 2001).



To explore the planimeter, visit the URL http://ncd3.math.harvard.edu/ccp/materials/mvcalc/green/index.html

GENERAL FUNDAMENTAL THEOREM OF CALCULUS. Green's theorem is of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are d such theorems in dimensions d. In the plane, Green's theorem is the second integral theorem after the fundamental theorem of line integrals FTLI. In three dimensions, there are two more theorems beside the FTLI: Stokes and Gauss Theorems.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus
2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Gauss theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence