Homework for Tuesday, December 17, 2002: Section 13.7: 2, 10, 12, 14, 18

DEFINITIONS. The curl of a vector field F is

$$\mathrm{curl}(P,Q,R) = \nabla \times F = (R_y - Q_z, P_z - R_x, Q_x - P_y) \ .$$

The flux integral of a vector field F through a surface $S = X(R)$ is defined as

$$\int\int_R F(X(u,v)) \cdot X_u \times X_v \ dudv$$

The line integral of a vector field F along a curve $\gamma = r([a,b])$ is given as

$$\int_a^b F(r(t)) \cdot r'(t) \ dt \ .$$

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library.

STOKES THEOREM. Let S be a surface with boundary curve γ and let F be a vector field. Then

$$\int\int_S \mathrm{curl}(F) \ \cdot dS = \int_\gamma F \cdot ds \ .$$

Note: the orientation of γ is such that if you walk along the surface (head into the direction of the normal $X_u \times X_v$), then you have the surface to your left.

EXAMPLE. Let $F(x,y,z) = (-y,x,0)$ and let S be the upper semi hemisphere, then $\mathrm{curl}(F)(x,y,z) = (0,0,2)$. The surface is parameterized by $r(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ on $R = [0,2\pi] \times [0,\pi/2]$ and $r_u \times r_v = \sin(v) r(u,v)$ so that $\mathrm{curl}(F)(x,y,z) \cdot r_u \times r_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \ dvdu = 2\pi$.

The boundary γ of S is parameterized by $r(t) = (\cos(t), \sin(t), 0)$ so that $ds = r'(t)dt = (-\sin(t), \cos(t), 0)dt$ and $F(r(t)) \ r'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_\gamma F \cdot ds$ along the boundary is 2π.

SPECIAL CASE: GREEN'S THEOREM. If S is a surface in the $x-y$ plane and $F = (P,Q,0)$ has zero z component, then $\mathrm{curl}(F) = (0,0,Q_x - P_y)$ and $\mathrm{curl}(F) \cdot dS = (Q_x - P_y) \ dxdy$.

CONSEQUENCE. The flux of the curl of a vector field through a surface depends only on the boundary.

PROOF OF STOKES THEOREM. For a surface which is a parallelepiped Stokes theorem is seen with Green's theorem: the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\mathrm{curl}(F)$. For a general surface, we approximate the surface through a mesh of small parallelepipeds. When summing up line integrals along all these parallelepipeds, most cancel and only the integral along the boundary stays. The sum of the fluxes of the curl through boundary becomes the flux through the surface.

DIVERGENCE. The divergence of a vector field F is $\mathrm{div}(P,Q,R) = \nabla \cdot F = P_x + Q_y + R_z$. It is a scalar field.

The flux integral of a vector field F through a surface $S = X(R)$ is defined as $\int\int_S F \cdot dS = \int\int_R F(X(u,v)) \cdot X_u \times X_v \ dudv$.

The integral of a scalar function f on a region G is $\int\int\int_G f dV = \int\int\int_G f(x,y,z) \ dxdydz$.

GAUSS THEOREM (or divergence theorem). Let G be a body in space bounded by a surface S and let F be a vector field. Then

$$\int\int\int_G \mathrm{div}(F) \ dV = \int\int_S F \ \cdot dS$$

Note: the orientation of S is such that the normal vector $X_u \times X_v$ points outside of G.

EXAMPLE. Let $F(x,y,z) = (x,y,z)$ and let S be sphere. The divergence of F is 3 and $\int\int\int_G \text{div}(F)\ dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S X \cdot X_u \times X_v\ dudv = \int\int_S |X(u,v)|^2 \sin(v)\ dudv = \int_0^\pi \int_0^{2\pi} \sin(v)\ dudv = 4\pi$.

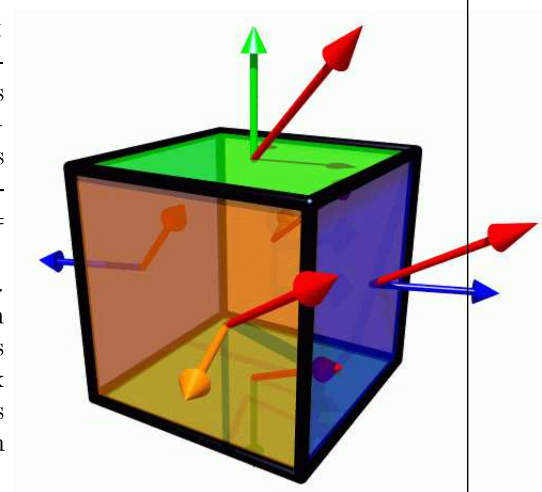
APPLICATION: CONTINUITY EQUATION. If ρ is the density of a fluid and v is the velocity field of the fluid, then by conservation of mass, the flux $\int\int_S v \cdot dS$ of the fluid through a closed surface S bounding a region G should be $d/dt \int\int\int_G \rho dV$, the change of mass inside G. But this flux is by Gauss theorem equal to $\int\int\int_G \text{div}(v) dV$. Therefore, $\int\int\int_G \dot{\rho} - \text{div}(v)\ dV = 0$. Taking a very small ball G around a point (x,y,z), where $\int\int\int_G f\ dV \sim f(x,y,z)$ gives $\dot{\rho} = \text{div}(v)$. This is called the **continuity equation**.

EXAMPLE. What is the flux of the vector field $F(x,y,z) = (2x, 3z^2 + y, \sin(x))$ through the box $G = [0,3] \times [0,2] \times [-1,1]$?

ANSWER. Use the divergence theorem: $\text{div}(F) = 2$ and $\int\int\int_G \text{div}(F)\ dV = 2\int\int\int_G dV = 2\text{Vol}(G) = 24$.

NOTE: Often, it is easier to evaluate a three dimensional integral than a flux integral because the later needs a parameterization of the boundary, calculation of $X_u \times X_v$ etc.

PROOF OF GAUSS THEOREM. Consider a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. Call the sides orthogonal to the x axes **x-boundaries** etc. The flux of $F = (P,Q,R)$ through the x-boundaries is $[F(x+dx,y,z) \cdot (1,0,0) + F(x,y,z) \cdot (-1,0,0)]dydz = P(x+dx,y,z) - P(x,y,z) = P_x dxdydz$. Similarly, the flux through the y-boundaries is $P_y dydxdz$ and the flux through the z-boundary is $P_z dzdxdy$. The total flux through the boundary of the box is $(P_x + P_y + P_z)dxdydz = \text{div}(F)dxdydz$.
For a general body, approximate it with a union of small little cubes. The sum of the fluxes over all the little cubes is sum of the fluxes through the sides which do not touch an other box (fluxes through touching sides cancel). The sum of all the infinitesimal fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\text{div}(F)dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \text{div}(F)dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain Gauss theorem.



VOLUME CALCULATION. Similarly as the area of a region can be computed using Green's theorem, the volume of a region can be determined as a flux integral.
Take for example the vector field $F(x,y,z) = (x,0,0)$ which has divergence 1. The flux of this vector field through the boundary of a region is the volume of the region. $\int\int_{\delta G} (x,0,0) \cdot dS = \text{Vol}(G)$.

STOKES AND GAUSS. **Stokes theorem was found by Ampere in 1825.** George Gabriel Stokes: (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.
Gauss theorem was discovered 1764 by Joseph Louis Lagrange.
Carl Friedrich Gauss, who formulates also Greens theorem, rediscovers the divergence theorem in 1813.
Green also rediscovers the divergence theorem in 1825 not knowing of the work of Gauss and Lagrange.

Carl Friedrich Gauss

George Gabriel Stokes

Joseph Louis Lagrange

André Marie Ampere