

SUMMARY. We review the fundamental theorem of line integrals, Green's theorem, Stokes theorem and Gauss theorem. These four results generalize the fundamental theorem of calculus and are all of the form

$$\int_{dR} F = \int_R dF, \text{ where } dR \text{ is the boundary of } R \text{ and } dF \text{ is a derivative of } F.$$

INTEGRATION.

**Line integral:**  $\int_{\gamma} F \cdot ds = \int_a^b F(r(t)) \cdot r'(t) dt$

**Surface integral**  $\int \int_S f dS = \int_a^b \int_c^d f(X(u, v)) |X_u(u, v) \times X_v(u, v)| dudv$

**Flux integral:**  $\int \int_S F \cdot dS = \int_a^b \int_c^d F(X(u, v)) \cdot X_u(u, v) \times X_v(u, v) dudv$

**Double integral:**  $\int \int_R f dA = \int_a^b \int_c^d f(x, y) dx dy.$

**Triple integral:**  $\int \int \int_R f dV = \int_a^b \int_c^d \int_o^p f(x, y, z) dx dy dz.$

**Area**  $\int \int_R 1 dA = \int \int_R 1 dx dy =$

**Surface area**  $\int \int 1 dS =$

$\int \int |X_u \times X_v| dudv$

**Volume**  $\int \int \int_B 1 dV =$

$\int \int \int 1 dx dy dz$

DIFFERENTIATION.

**Derivative:**  $f'(t) = \dot{f}(t) = d/dt f(t).$

**Partial derivative:**  $U_x(x, y, z) = \frac{\partial U}{\partial x}(x, y, z).$

**Gradient:**  $\text{grad}(U) = (U_x, U_y, U_z)$

**Curl in 2D:**  $\text{curl}(F) = \text{curl}((P, Q)) = Q_x - P_y$

**Curl in 3D:**  $\text{curl}(F) = \text{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$

**Div:**  $\text{div}(F) = \text{div}(P, Q, R) = P_x + Q_y + R_z.$

IDENTITIES.

$\text{div}(\text{curl}(F)) = 0$

$\text{curl}(\text{grad}(U)) =$

$(0, 0, 0)$

LINE INTEGRAL THEOREM. If  $\gamma : t \mapsto r(t), t \in [a, b]$  is a curve and  $U$  is a function either in 3D or the plane. Then

$$\int_{\gamma} \nabla U \cdot ds = r(b) - r(a)$$

where  $\nabla U = \text{grad}(U)$  is the gradient of  $U$ .

CONSEQUENCES.

1) If the curve is closed, then the line integral  $\int_{\gamma} \nabla U \cdot ds$  is zero.

2) If  $F = \nabla U$ , the line integral between two points  $P$  and  $Q$  does not depend on the chosen path.

REMARKS.

1) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus**

$$\int_a^b U'(x) dx = U(b) - U(a)$$

2) The theorem justifies the name **conservative** for gradient vector fields.

3) In physics,  $U$  is the **potential energy** and  $\nabla U$  a **force**. The theorem says that for such forces we have **energy conservation**.

PROBLEM. Let  $U(x, y, z) = x^2 + y^4 + z$ . Find the line integral of the vector field  $F(x, y, z) = \nabla U(x, y, z)$  along the path  $r(t) = (\cos(5t), \sin(2t), t^2)$  from  $t = 0$  to  $t = 2\pi$ .

SOLUTION.  $r(0) = (1, 0, 0)$  and  $r(2\pi) = (1, 0, 4\pi^2)$  and  $U(r(0)) = 1$  and  $U(r(2\pi)) = 1 + 4\pi^2$ . Applying the fundamental theorem gives  $\int_{\gamma} \nabla U ds = U(r(2\pi)) - U(r(0)) = 4\pi^2$ .

GREEN'S THEOREM. If  $R$  is a region with boundary  $\gamma$  and  $F = (P, Q)$  is a vector field, then

$$\int \int_R \text{curl}(F) dA = \int_{\gamma} F \cdot ds$$

where  $\text{curl}(F)(x, y) = Q_x - P_y$ .

REMARKS.

1) The theorem can be used to calculate two dimensional integrals by a one dimensional integral along a curve. Sometimes, however, it is used to evaluate line integrals by evaluating 2D integrals.

2) The curve is oriented in such a way that the region is to your left.

3) The region has to have piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set).

4) If  $\gamma : t \mapsto r(t) = (x(t), y(t))$ , the line integral is  $\int_a^b (P(x(t), y(t)), Q(x(t), y(t))) \cdot (x'(t), y'(t)) dt$ .

5) Green's theorem was found by George Green (1793-1841) in 1827 and by Michel Ostogradski (1801-1861).

**CONSEQUENCES.**

- 1) If  $\text{curl}(F) = 0$  everywhere in space or the plane, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
- 2) Taking  $F(x, y) = (-y, 0)$  gives an **area formula**  $\text{Area}(R) = \int -y \, dx$ . Similarly  $\text{Area}(A) = \int x \, dy$ .

**PROBLEM.** Find the line integral of the vector field  $F(x, y) = (x^4 + \sin(x) + y, x + y^3)$  along the path  $r(t) = (\cos(t), 5 \sin(t) + \log(1 + \sin(t)))$ , where  $t$  runs from  $t = 0$  to  $t = \pi$ .

**SOLUTION.**  $\text{curl}(F) = 0$  implies that the line integral depends only on the end points  $(0, 1), (0, -1)$  of the path. Take the simpler path  $r(t) = (-t, 0), t = [-1, 1]$ , which has velocity  $r'(t) = (-1, 0)$ . The line integral is  $\int_{-1}^1 (t^4 - \sin(t), -t) \cdot (-1, 0) \, dt = -t^5/5|_{-1}^1 = -2/5$ .

**REMARK.** We could also find a potential  $U(x, y) = x^5/5 - \cos(x) + xy + y^5/4$ . It has the property that  $\text{grad}(U) = F$ . Again, we get  $U(0, -1) - U(0, 1) = -1/5 - 1/5 = -2/5$ .

**STOKES THEOREM.** If  $S$  is a surface in space with boundary  $\gamma$  and  $F$  is a vector field, then

$$\int \int_S \text{curl}(F) \cdot dS = \int_{\gamma} F \cdot ds$$

**REMARKS.**

- 1) Stokes theorem reduces to Greens theorem if  $F$  is  $z$  independent and  $S$  is contained in the  $z$ -plane.
- 2) The orientation of  $\gamma$  is such that if you walk along  $\gamma$  and have your head in the direction, where the normal vector  $X_u \times X_v$  of  $S$  points, then you have the surface to your left.
- 3) Stokes theorem was found by André Ampère (1775-1836) in 1825. It was rediscovered by George Stokes (1819-1903).

**CONSEQUENCES.**

- 1) The flux of the curl of a vector field does not depend on the surface  $S$ , only on the boundary of  $S$ . This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
- 2) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

**PROBLEM.** Compute the line integral of  $F(x, y, z) = (x^3 + xy, y, z)$  along the polygonal path  $\gamma$  connecting the points  $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$ .

**SOLUTION.** The path  $\gamma$  bounds a surface  $S : X(u, v) = (u, v, 0)$  parameterized by  $R = [0, 2] \times [0, 1]$ . By Stokes theorem, the line integral is equal to the flux of  $\text{curl}(F)(x, y, z) = (0, 0, -x)$  through  $S$ . The normal vector of  $S$  is  $X_u \times X_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$  so that  $\int \int_S \text{curl}(F) \, dS = \int_0^2 \int_0^1 (0, 0, -u) \cdot (0, 0, 1) \, dudv = \int_0^2 \int_0^1 -u \, dudv = -2$ .

**GAUSS THEOREM.** If  $S$  is the boundary of a region  $B$  in space with boundary  $S$  and  $F$  is a vector field, then

$$\int \int \int_B \text{div}(F) \, dV = \int \int_S F \cdot dS$$

where  $\text{div}(F)$  is the divergence of  $F$ .

**REMARKS.**

- 1) Gauss theorem is also called **divergence theorem**.
- 2) Gauss theorem can be helpful to determine the flux of vector fields through surfaces.
- 3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.

**CONSEQUENCES.**

- 1) For divergence free vector fields  $F$ , the flux through a closed surface is zero. Such fields  $F$  are also called **incompressible** or **source free**.

**PROBLEM.** Compute the flux of the vector field  $F(x, y, z) = (-x, y, z^2)$  through the boundary  $S$  of the rectangular box  $[0, 3] \times [-1, 2] \times [1, 2]$ .

**SOLUTION.** By Gauss theorem, the flux is equal to the triple integral of  $\text{div}(F) = 2z$  over the box:  $\int_0^3 \int_{-1}^2 \int_1^2 2z \, dx \, dy \, dz = (3 - 0)(2 - (-1))(4 - 1) = 27$ .

SUMMARY. These two pages give some information about people who were involved in the discovery of the integral theorems.

AMPÈRE. André Marie Ampère (1775-1836) made contributions to the theory of electricity and magnetism. Stokes theorem was found by Ampère in 1825.



Gossip:

- While still only 13 years old Ampère submitted his first paper to the Académie de Lyon. Amperès father died during the French Revolution. He went to the guillotine writing to Ampère's mother from his cell: "I desire my death to be the seal of a general reconciliation between all our brothers; I pardon those who rejoice in it, those who provoked it, and those who ordered it.... "
- Ampère had a difficult time with his daughter. She married one of Napoleon's lieutenants who was an alcoholic. The marriage soon was in trouble. Ampère's daughter fled to her father's house and Ampère allowed later her husband to live with him also. This proved a difficult situation, led to police intervention and much unhappiness for Ampère.

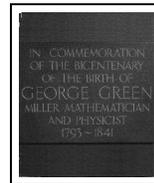
GAUSS. Carl Friedrich Gauss (1777-1855) worked in a wide variety of fields both in mathematics and physics like number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. Gauss theorem was discovered 1764 by Joseph Louis Lagrange.



Gossip:

- At the age of seven, while starting elementary school his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed up the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101.
- Gauss married twice. Both wives died, the first from the birth of a child, the second after a long illness.

GREEN. George Green (1793-1841) discovered Green's theorem in 1827. Green also rediscovers the divergence theorem in 1825 not knowing of the work of Gauss and Lagrange.



Gossip:

- Green studied mathematics on the top floor of his fathers mill, entirely on his own. The years between 1823 and 1828 were not easy for Green: As well as having a full time job in the mill, two daughters were born, the one in 1824 mentioned above, and a second in 1827. Between these two events his mother had died in 1825 and his father was to die in 1829. Yet despite the difficult circumstances and despite his flimsy mathematical background, Green published one of the most important mathematical works of all time in 1828.
- Green was the son of a baker: from a short obituary in the Nottingham Review which showed they knew little of his life and less of the importance of his work. "... Through Thomson, Maxwell, and others, the general mathematical theory of potential developed by an obscure, self-taught millers son would lead to the mathematical theories of electricity underlying twentieth-century industry.

LAGRANGE. Joseph-Louis Lagrange (1736-1813) worked in all fields of analysis and number theory and analytical and celestial mechanics. Gauss theorem was discovered in 1764 by Joseph Louis Lagrange.



Gossip:

- Lagrange is usually considered a French mathematician. But some refer to him as an Italian mathematician because Lagrange was born in Turin. Only the intervention of Lavoisier saved Lagrange during the French revolution from the guillotine because a law was established arresting all foreign born people. Lavoisier himself was guillotined shortly after.
- Lagrange first wife was his cousin. They had no children. In fact, Lagrange had told d'Alembert in this letter that he did not wish to have children. His first wife died after years of illness.
- Napoleon named Lagrange to the Legion of Honor and Count of the Empire in 1808. On April 3, 1813 Lagrange was named "grand croix of the Ordre Impérial de la Réunion". He died a week later.

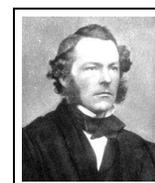
OSTROGRADSKI. Mikhail Vasilevich Ostrogradski (1801-1862) worked on integral calculus, mathematical physics like hydrodynamics and partial differential equations. He worked also in algebra and differential equations, theoretical mechanics and wrote several textbooks.



Gossip:

- Ostrogradski wanted actually to pursue a military carrier. Financial considerations however led him to a career in the civil service which needed a university education. He studied physics and mathematics. He successfully finished his exams but the minister of religious affairs and national education refused to confirm the decision. Ostrogradski therefore never got a degree.
- He left Russia to study in Paris, between 1822 and 1827 Ostrogradski attended lectures by Laplace, Fourier, Legendre, Poisson and Cauchy.

STOKES. George Gabriel Stokes (1819-1903) established the science of hydrodynamics with his law of viscosity describing the velocity of a small sphere through a viscous fluid. Stokes discovery of Stokes theorem (around 1840) was probably inspired by work of Green.



Gossip:

- In 1857 Stokes had to give up his fellowship at Pembroke College because fellows at Cambridge had then to be unmarried. Later, after a change in the rules in 1862 Stokes was able to take up the fellowship at Pembroke again.
- While Stokes became engaged, he used to express his feelings to his bride Mary Susanna Robinson in rather mathematical terms: "I too feel that I have been thinking too much of late, but in a different way, my head running on divergent series, the discontinuity of arbitrary constants ... ". Such words did not reveal the love that Mary hoped to find in them and when Stokes wrote her a 55 page (!) letter about the duty he felt towards her, she came close to calling off the wedding.

SOURCES. The sources include E.T. Whittaker's, *A History of the Theories of Ether and Electricity* and the MacTutor History of Mathematics <http://turnbull.dcs.st-and.ac.uk/history/>.