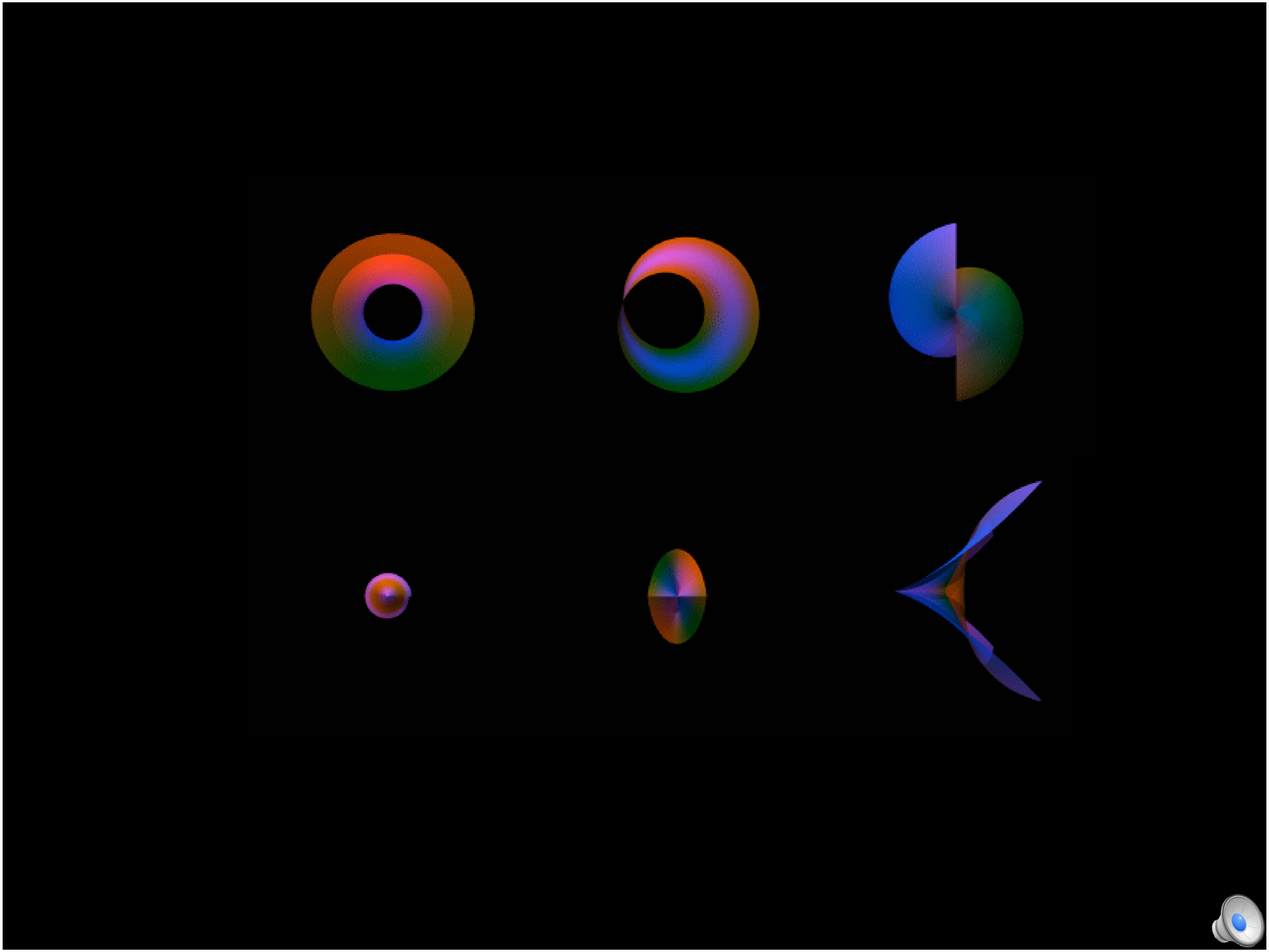
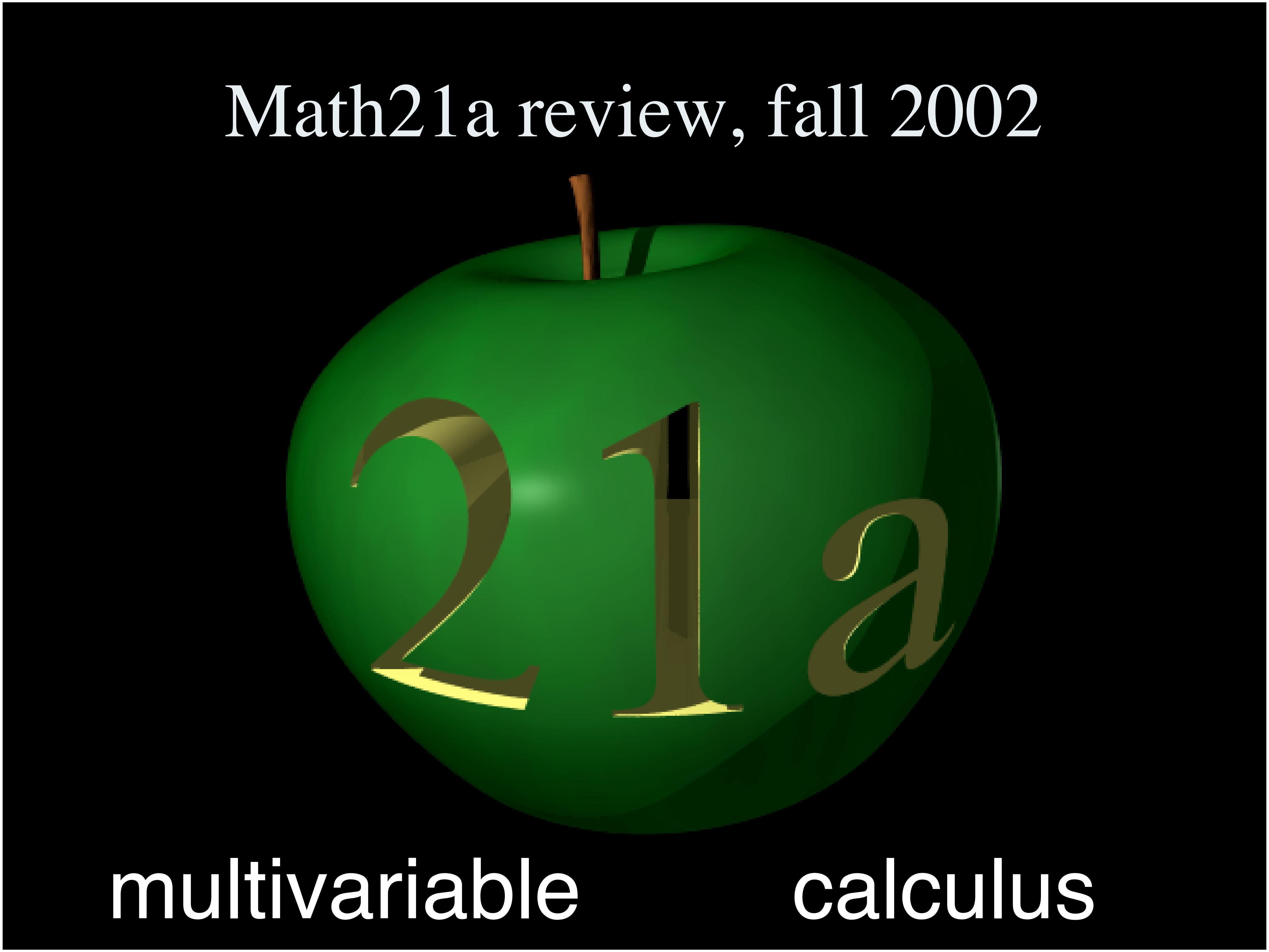



Math21a review, fall 2002
21a
multivariable
calculus

Math 21a Review

January 8, 2003

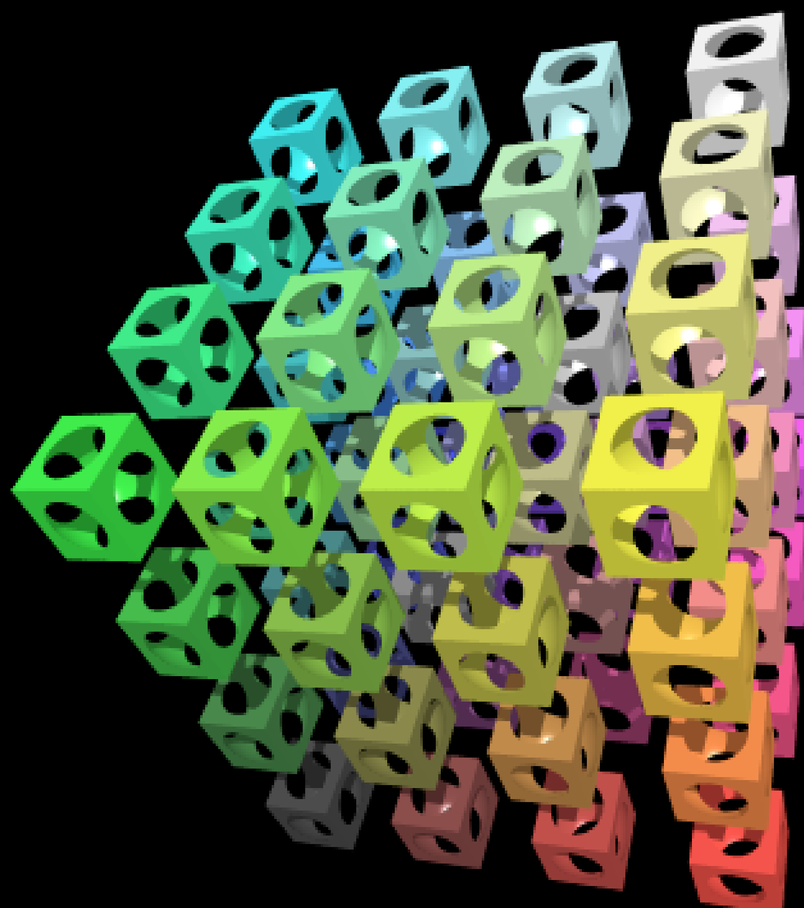
Oliver Knill

Topics

We review chapters 11 and 12

Topics

- Partial derivatives
- Linearization
- Chain rule
- Extrema
- Lagrange multipliers
- Double integrals
- Surface integrals
- Triple integrals



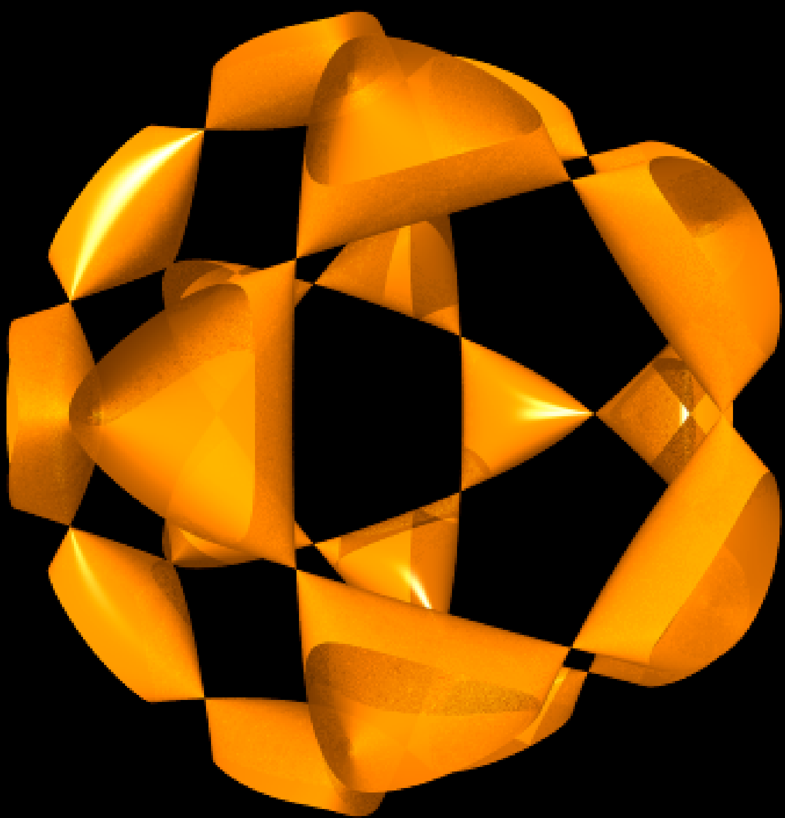
Watch for Army Knife Problems

Partial Derivatives

$$\frac{\partial f}{\partial x} = f_x$$

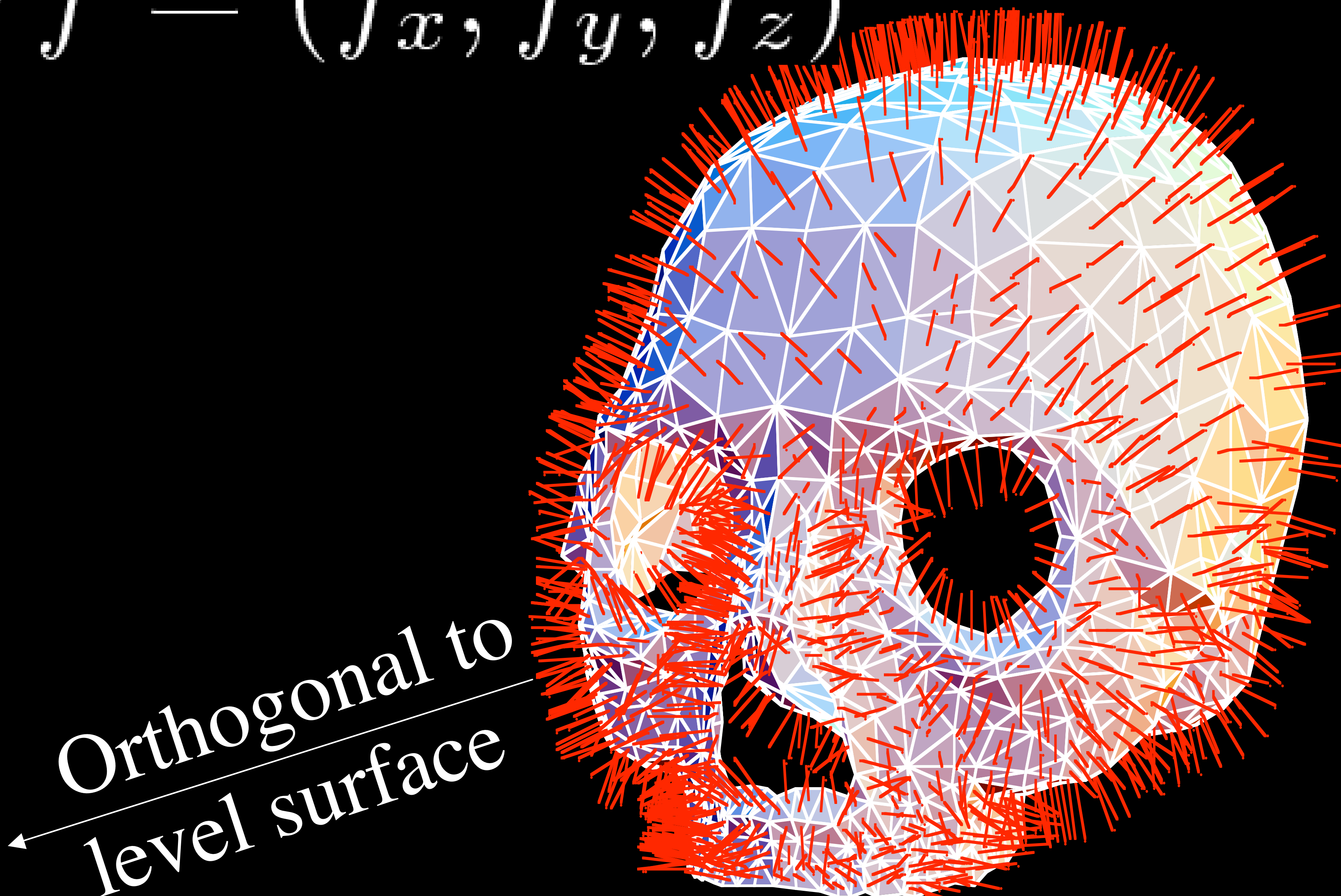
Partial Derivatives

- Directional derivative
- Gradient
- Curl
- Divergence
- Solutions of PDE's



Gradient

$$\nabla f = (f_x, f_y, f_z)$$



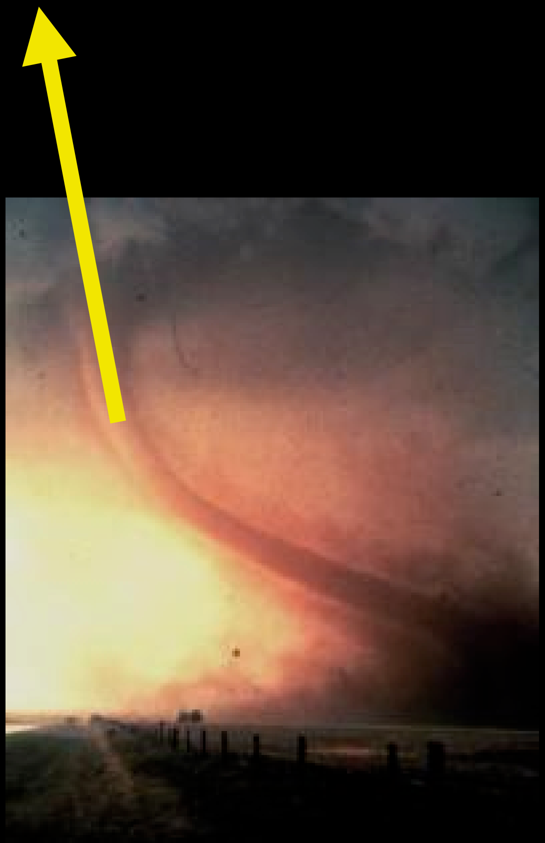
Directional Derivative

$$D_u f(x) = \nabla f(x) \cdot u$$

|u|=1

measures how f changes in the direction u.

Curl



$$F=(P,Q,R)$$

$$\mathrm{curl}(F) =$$

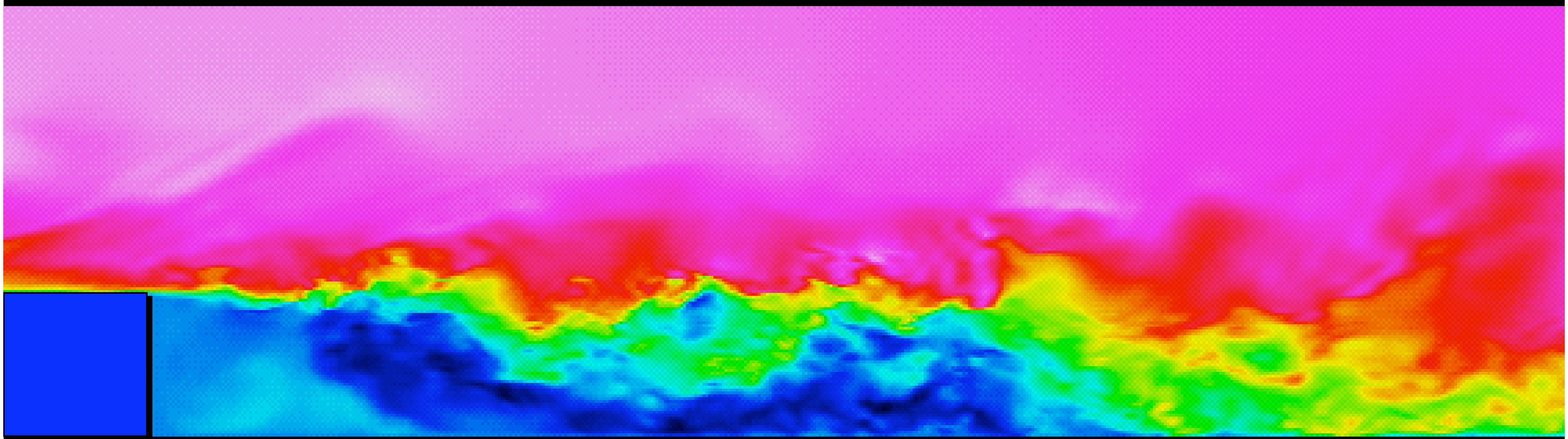
$$(R_y-Q_z,P_z-R_x,Q_x-P_y)$$

PDE example

$$\frac{d}{dt}u + u \cdot \nabla u = \nu \Delta u - \nabla p + f$$

$$\mathrm{div} u = 0$$

Navier Stokes

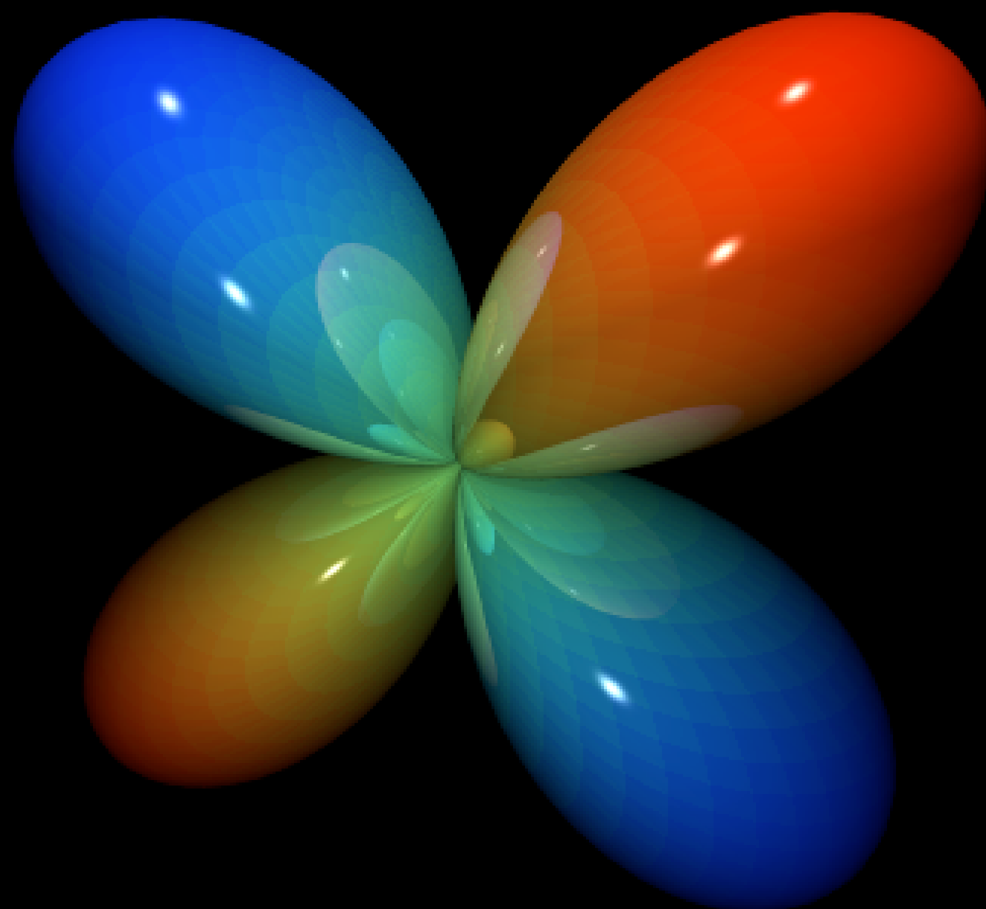


Flow behind obstacle

Problem 1

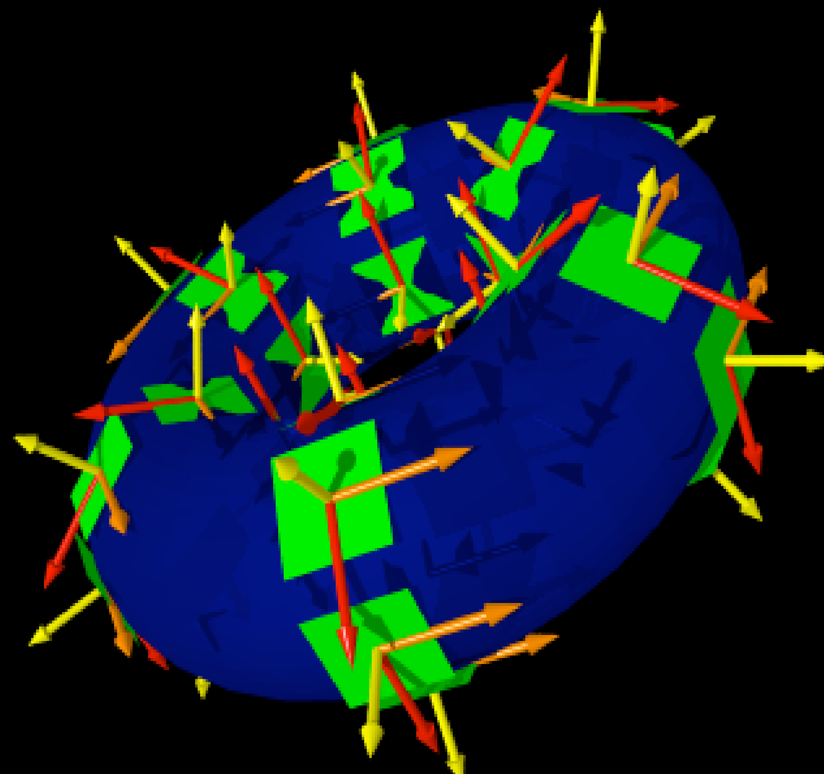
?

Linearization



Linearization

- Find tangent plane
- Linearization near a point
- Estimation

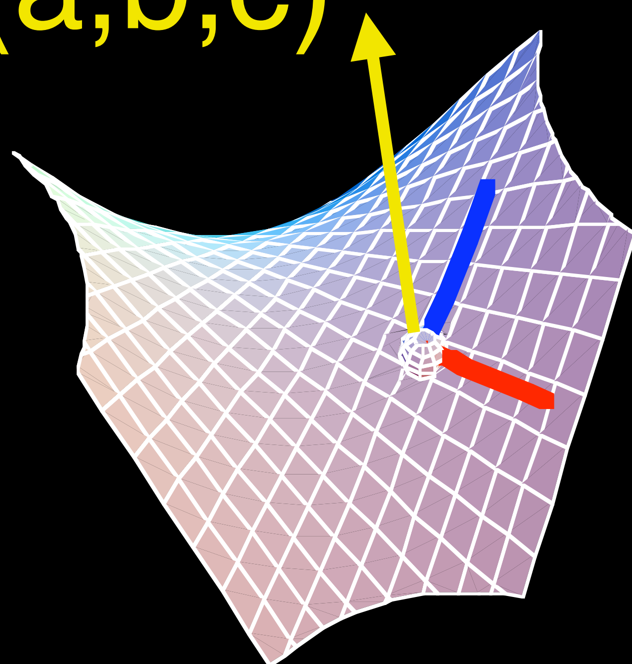


Tangent Plane

$n = grad(f)$

$n = |r_u \times r_v|$

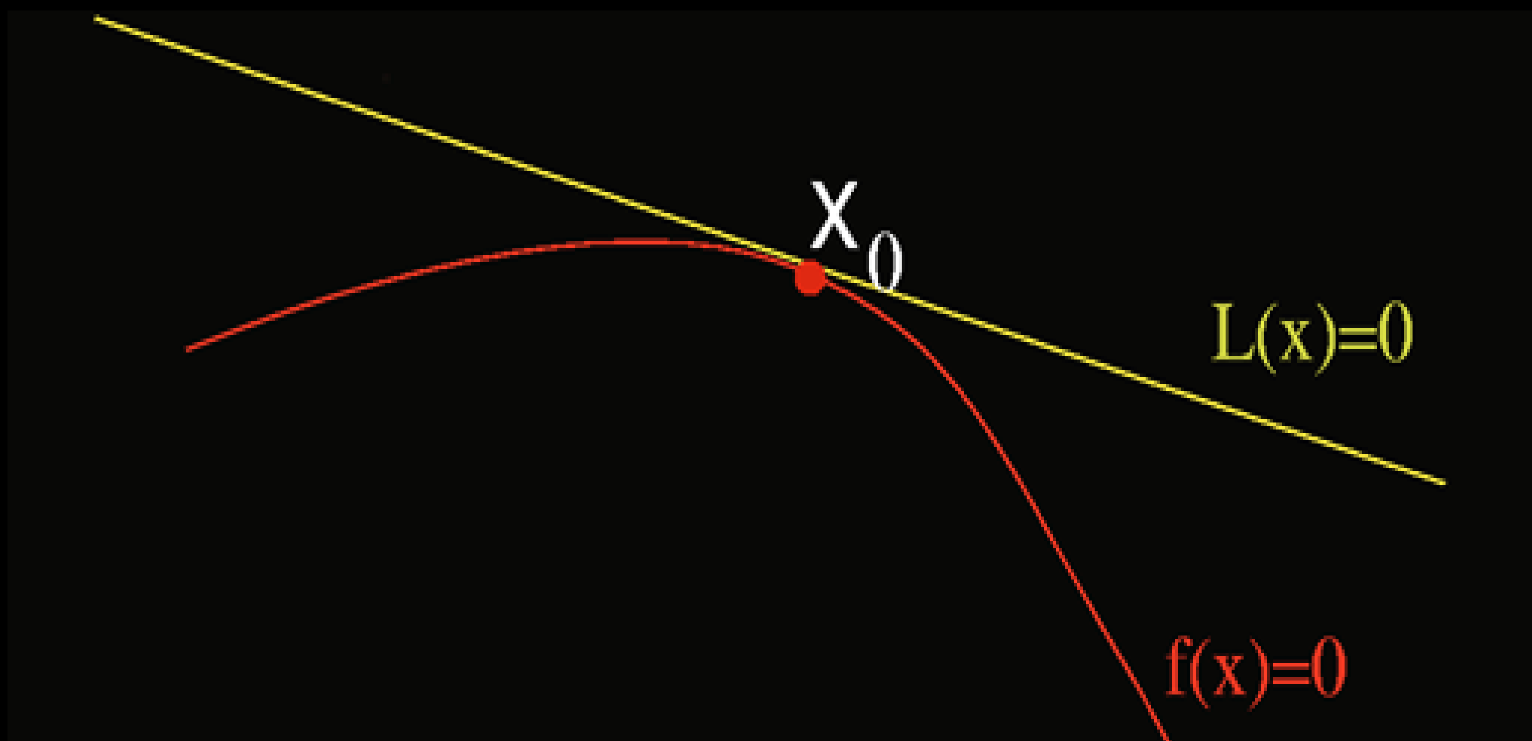
$n = (a,b,c)$



$ax + by + cz = d$

Linearization

$$f(x) \sim f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$



Estimation

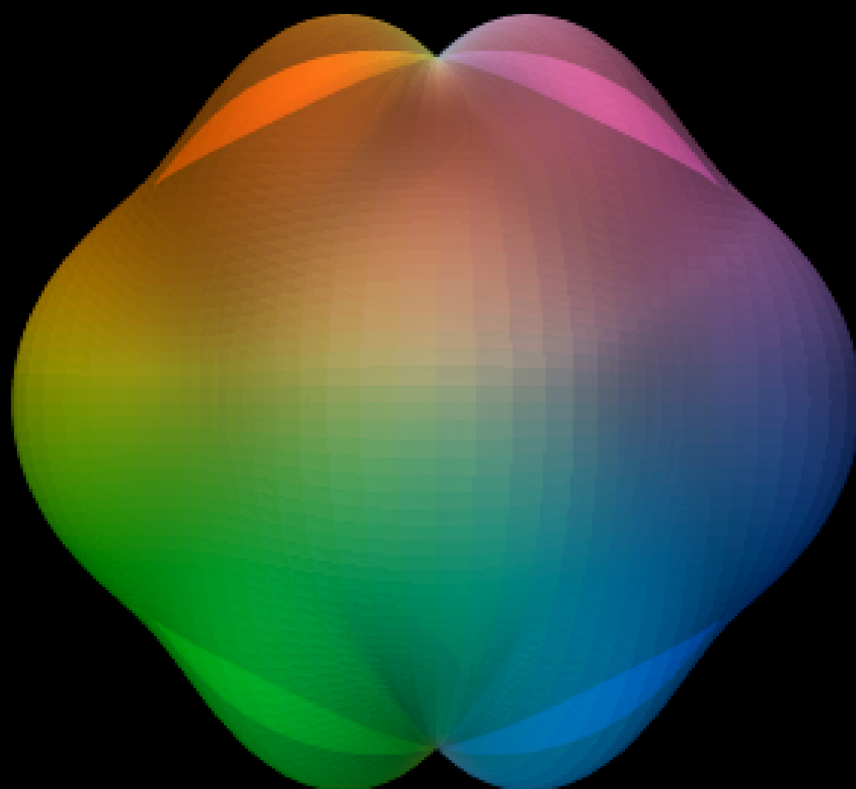
Estimate f(x) by L(x) near x_0

$$f(x,y)=\sin(x-2y)$$

$$\nabla f(0,0)=(1,-2)$$

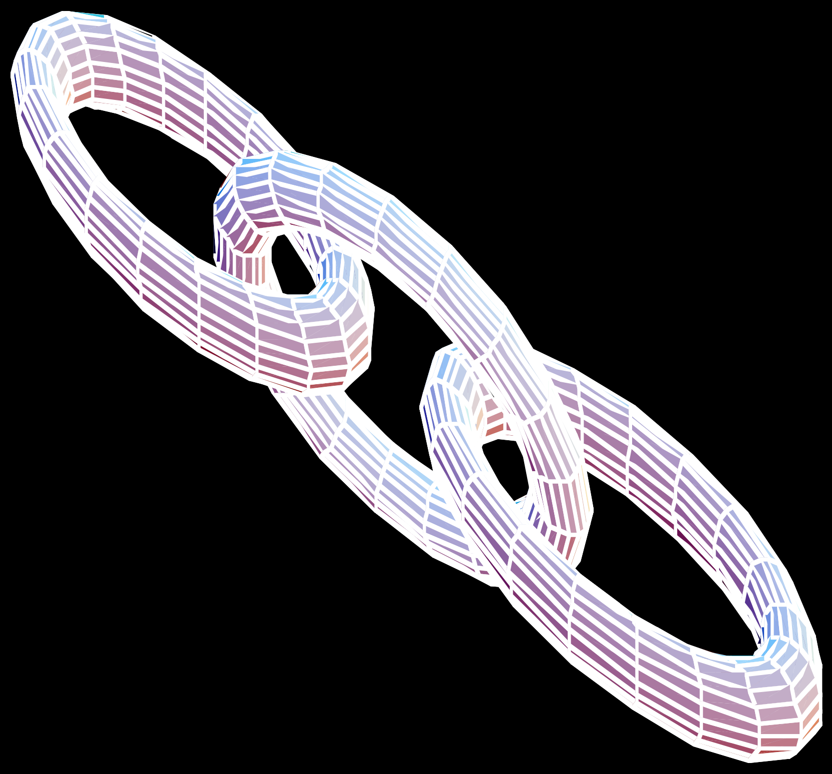
$$\sin(x-2y)\sim 0+x-2y$$

Chain Rule



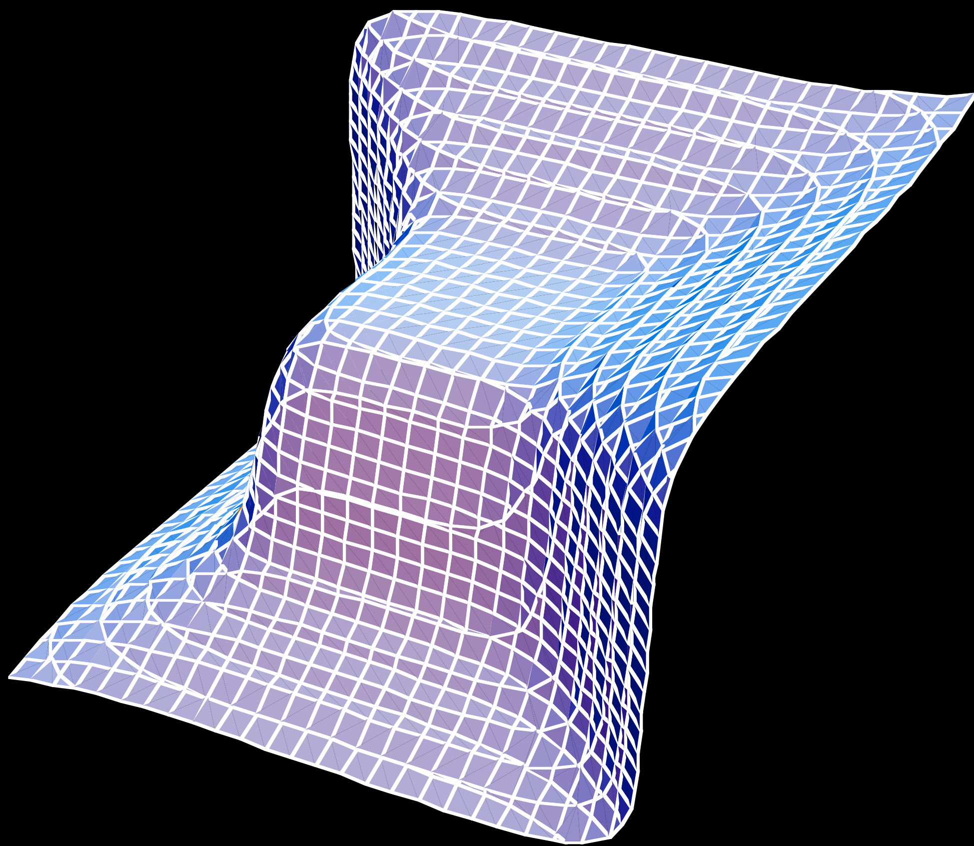
Chain Rule

- Form of 1D chain rule
- Implicit Derivatives
- Theoretical usage



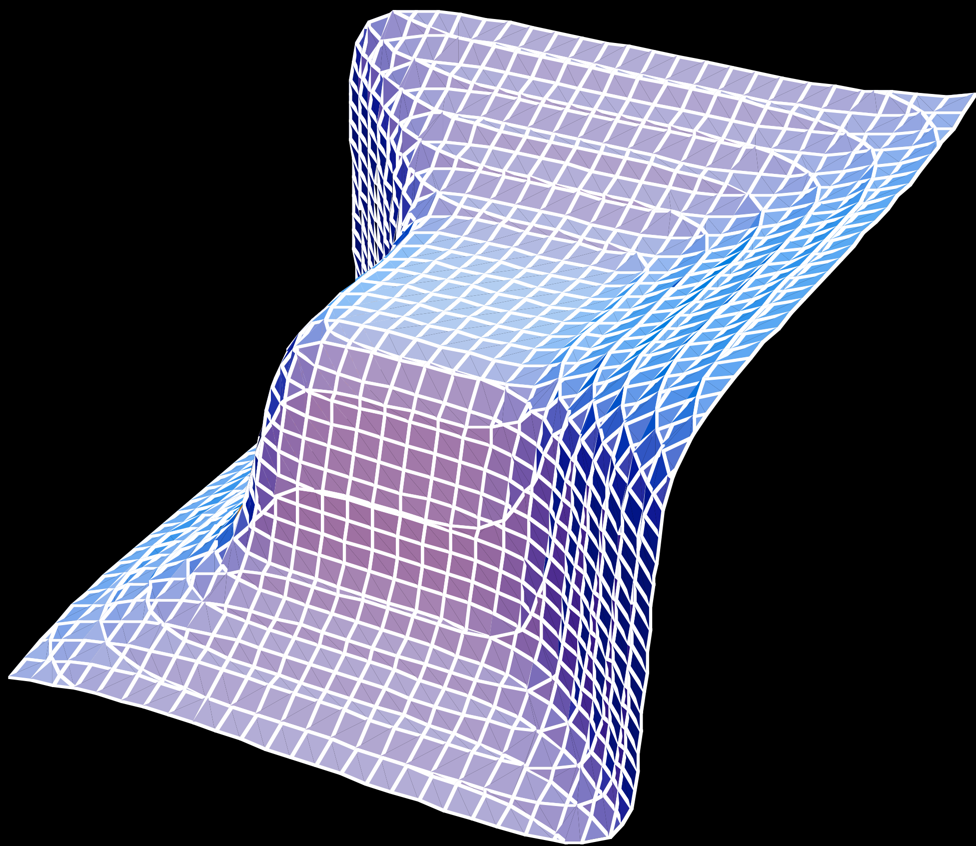
Chain Rule

$$\frac{d}{dt}f(r(t)) = \nabla f \cdot r'(t)$$



Implicit derivatives

$$x^5 + y^5 - z - z^7 - 1 = 0$$
$$z = g(x, y)$$
$$g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z) = -5x^4/(1 + 7z^6)$$
$$g_x(1, 0) = -5$$

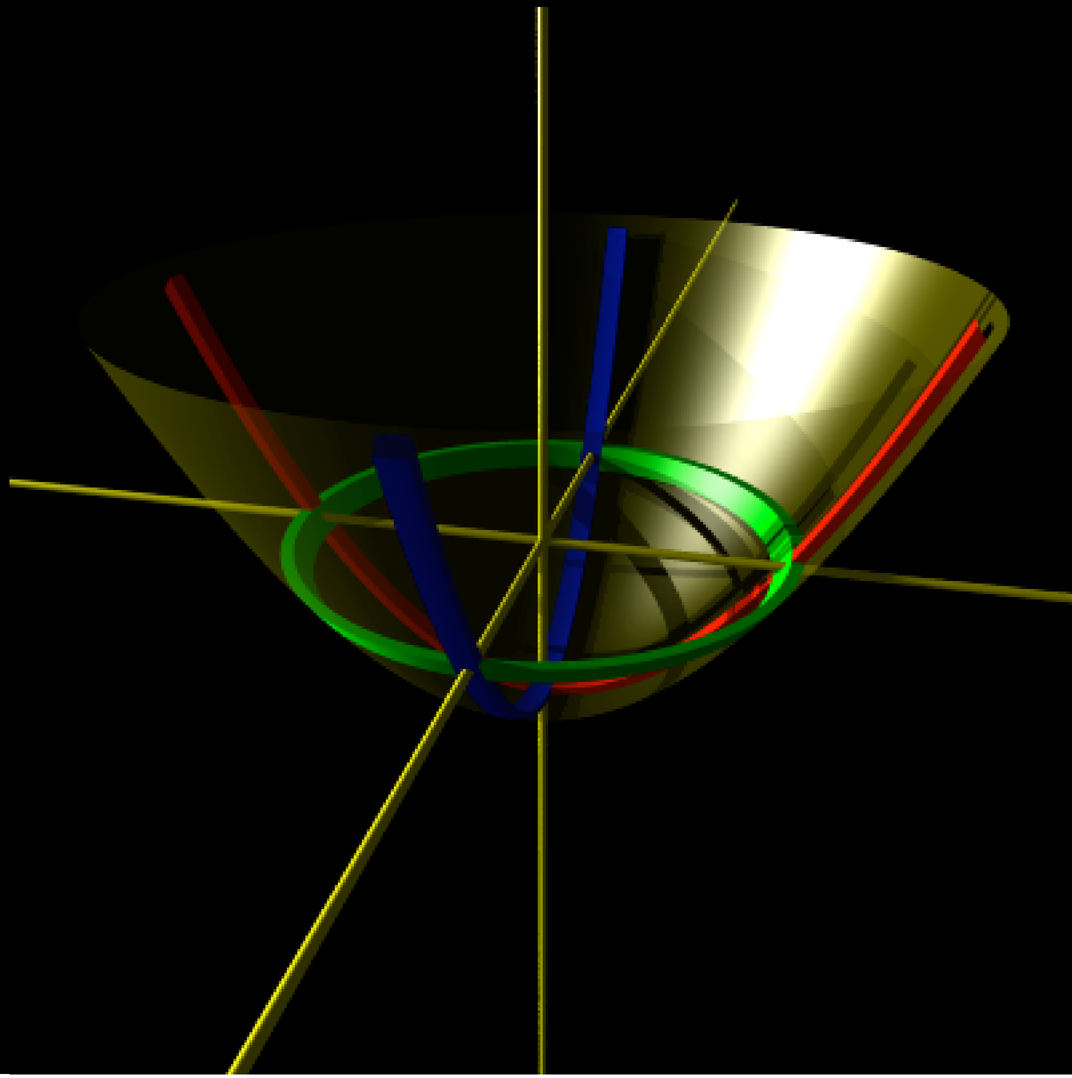


Implicit derivatives

$$f(x, y, g(x, y) = 0$$

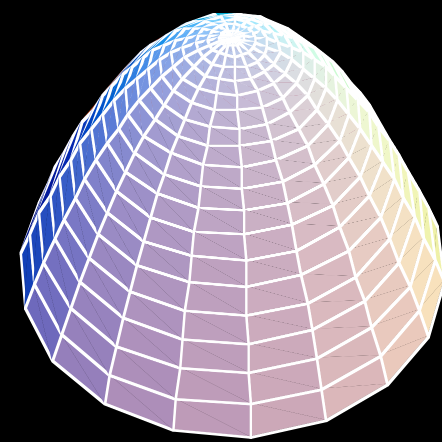
$$f_x(x, y, g(x, y)) + f_z(x, y, g(x, y)g_x(x, y) = 0.$$

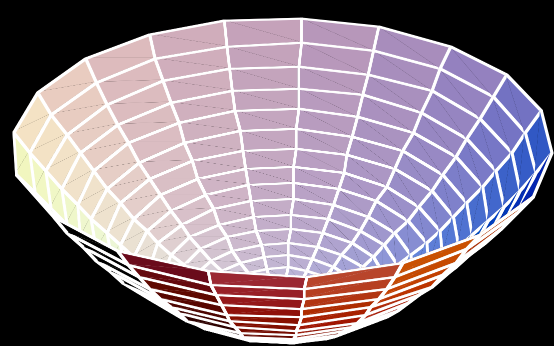
Maxima Minima

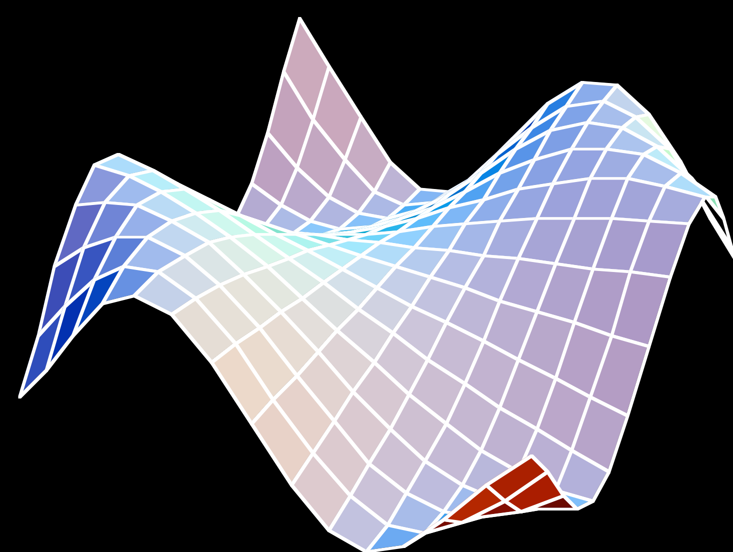


Optimization problems

- Identify critical points
- Second derivative test
- Identify maxima and minima



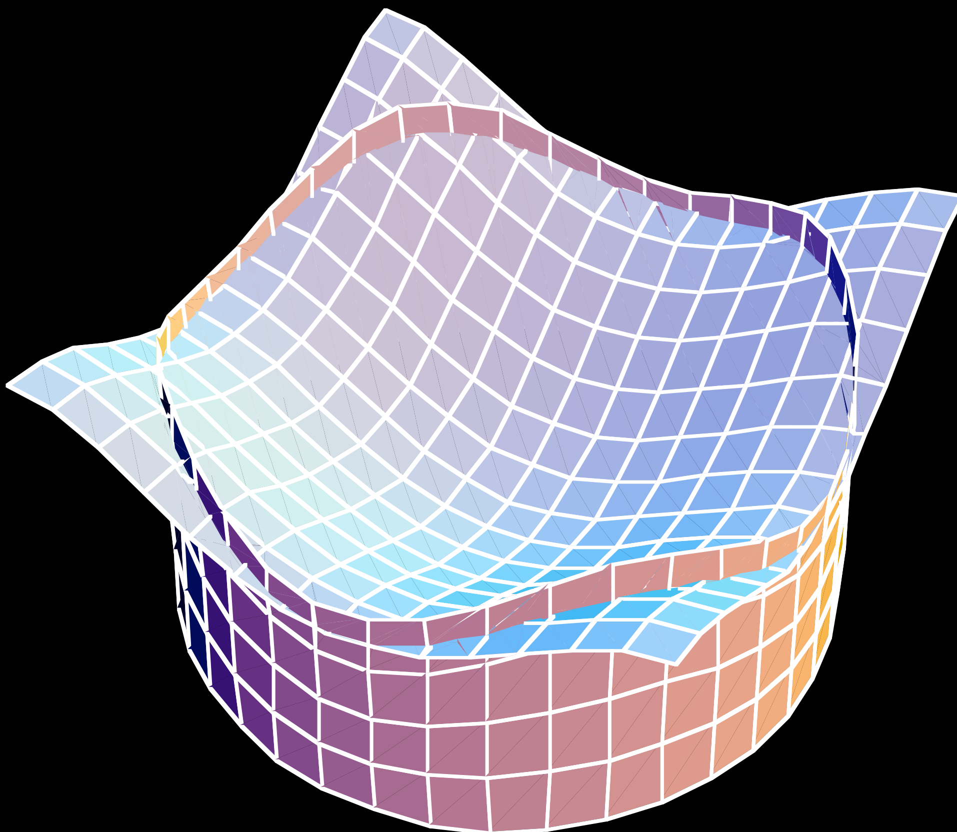




Second Derivative Test	
grad(f)=0	Critical Point
$D = f_{xx}f_{yy} - f_{xy}^2$	Discriminant
$D < 0$	Saddle
$D > 0, f_{xx} < 0$	Maximum
$D > 0, f_{xx} > 0$	Minimum

Example

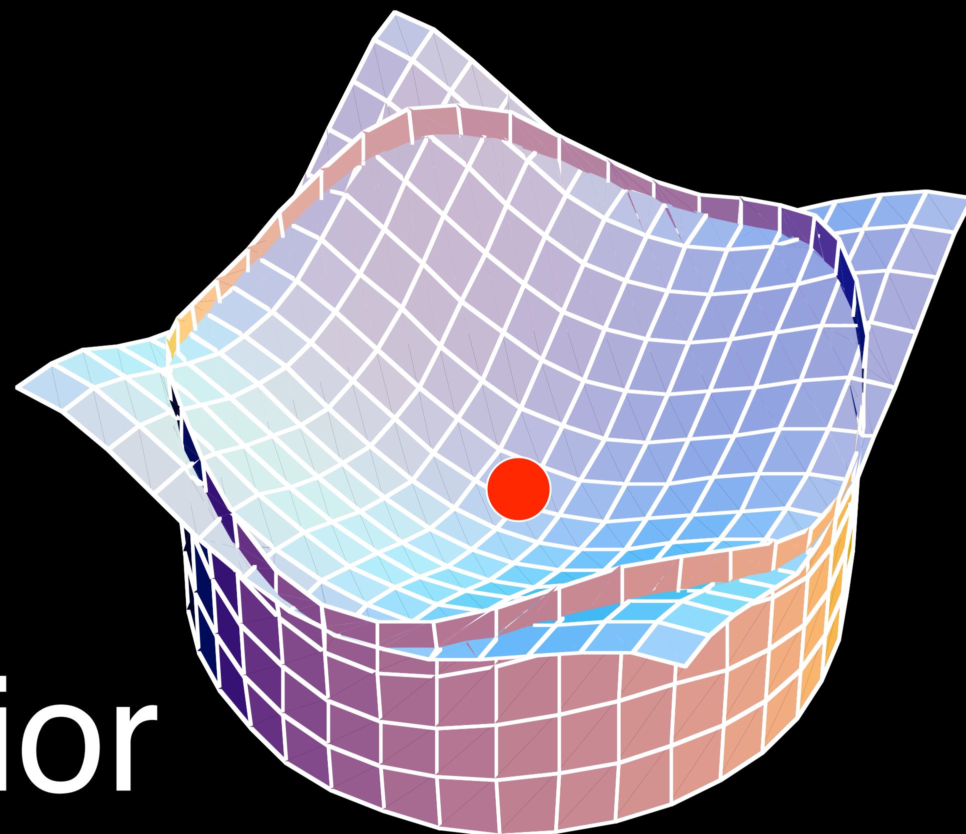
Find the extrema of the function



$$f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$$

Example

Critical points: only (0,0) In the interior



$$\nabla f(x,y) = (4x - 4x^3, 4y - 4y^3)$$

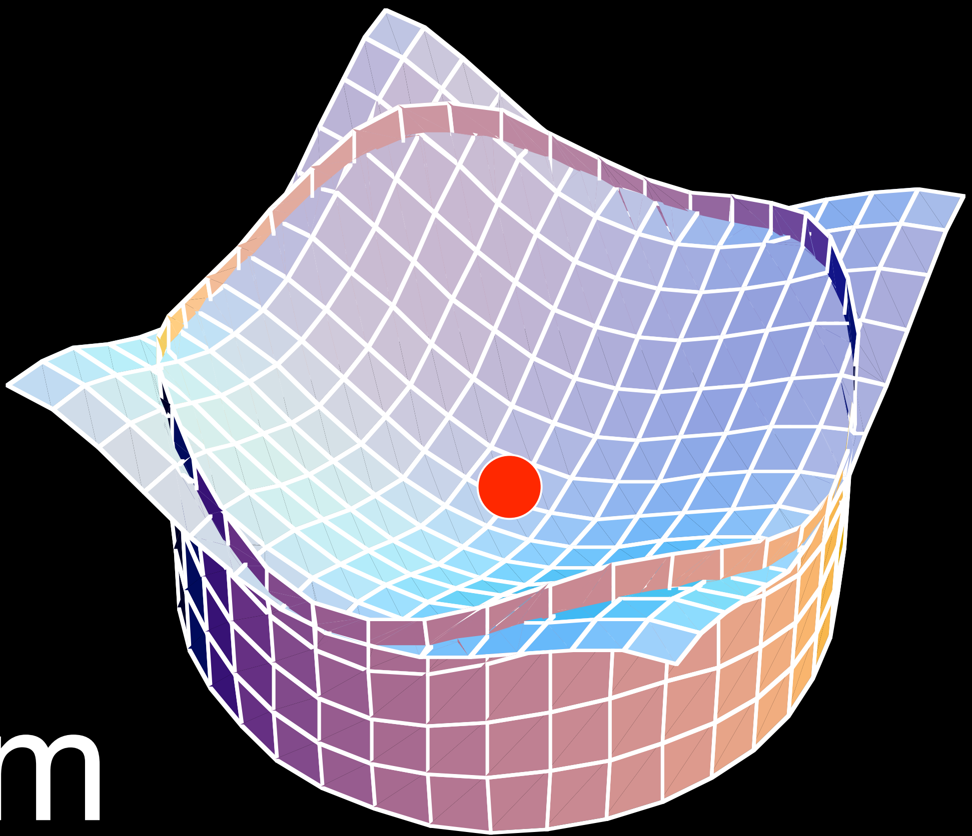
Example

$$f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$$

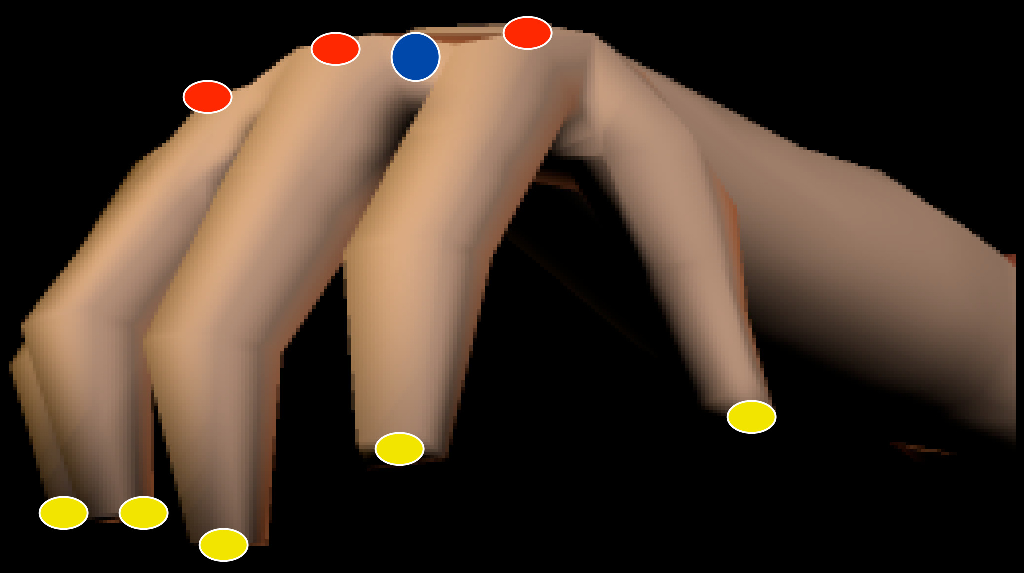
D=16> 0

$F_{xx} > 0$

local minimum



Critical points on Hand



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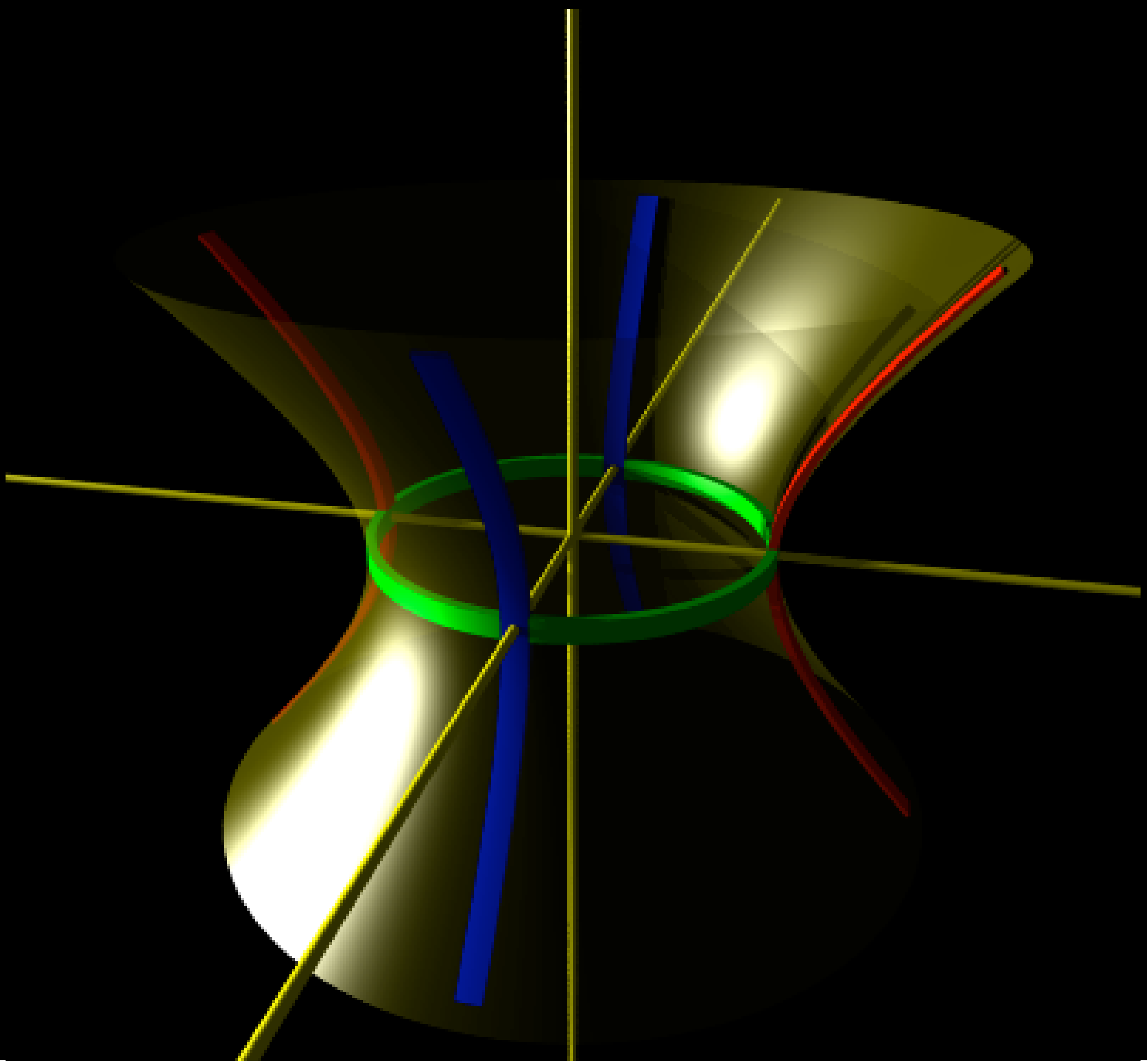
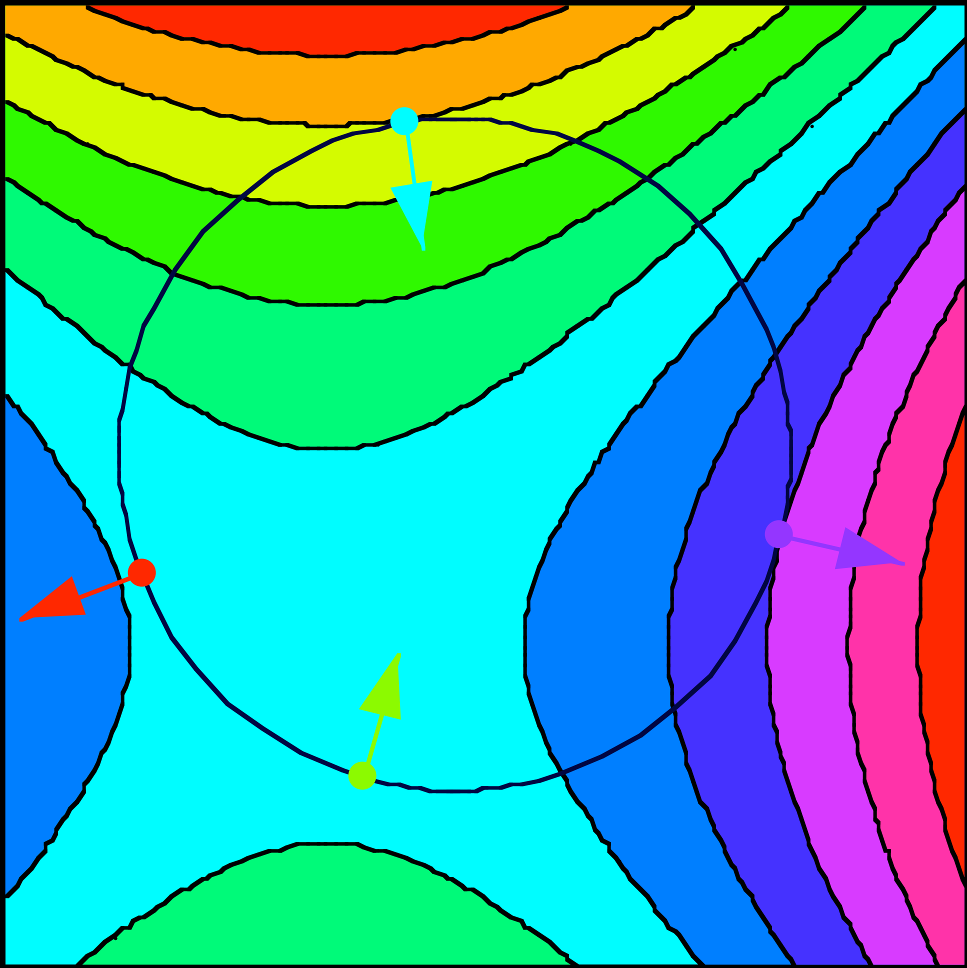
Problem 2

Lagrange Equations



Lagrange multipliers

- Lagrange equations
- Solving the system of equations
- Identifying maxima and minima



Setting up the equations

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= 0 \end{aligned}$$

Back to the example

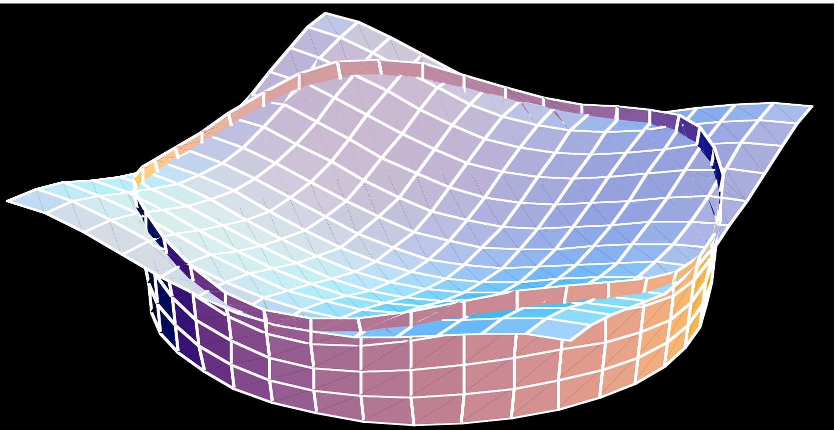
$$f(x, y) = 2x^2 + 2y^2 - x^4 - y^4$$

$$g(x, y) = x^2 + y^2 - 1$$

$$4x - 4x^3 = \lambda 2x$$

$$4y - 4y^3 = \lambda 2y$$

$$x^2 + y^2 = 1$$



$$2x(2 - \lambda - 2x^2) = 0$$

$$2y(2 - \lambda - 2y^2) = 0$$

$$x^2 + y^2 = 1$$

$$x = 0 \quad \text{or} \quad x = \pm y$$

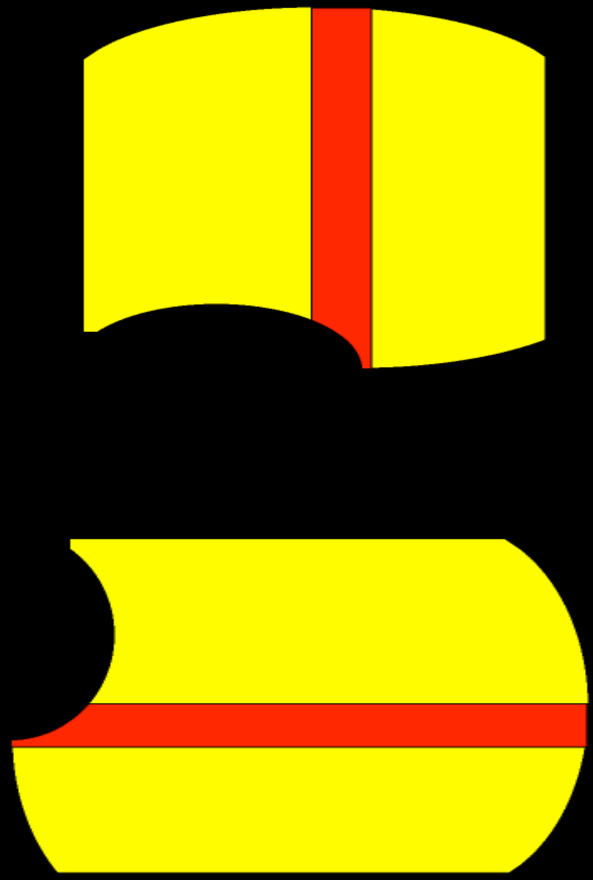
$$y = 0 \quad \text{or} \quad y = \pm y$$

$$x^2 + y^2 = 1$$

Double Integrals

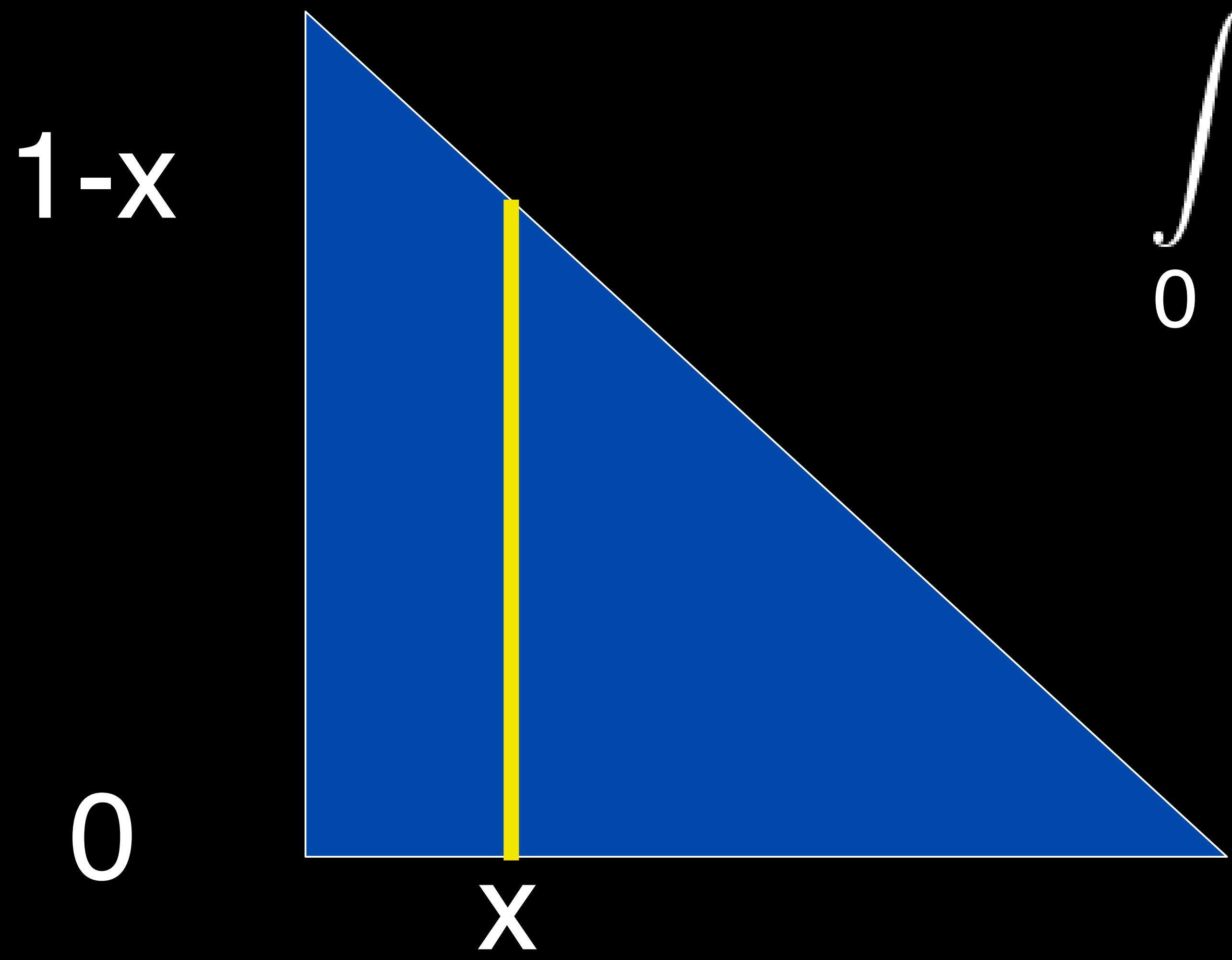
Double Integrals

- Type I regions
- Type II regions
- Polar coordinates
- Order of integration
- Area computation
- Make figure

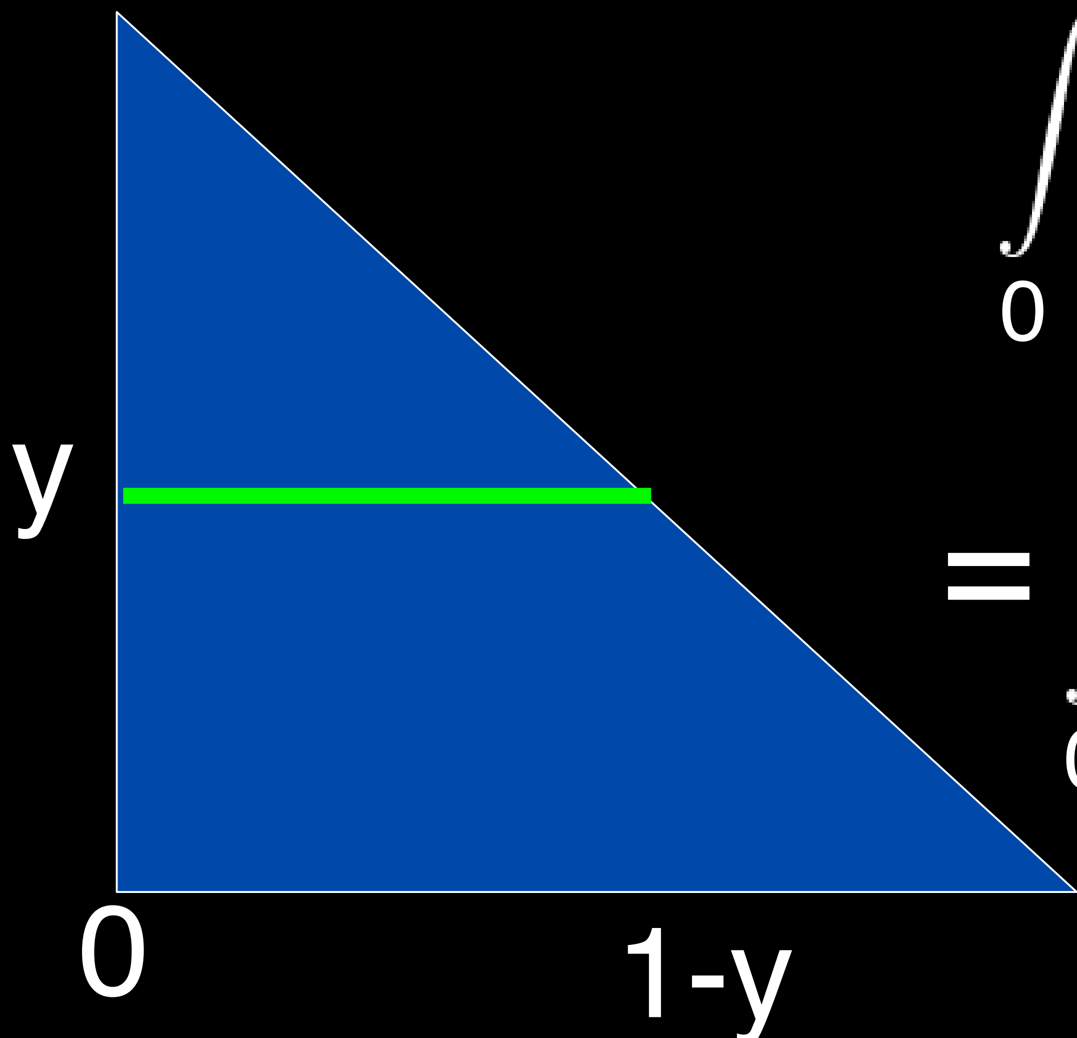


Change order of Integration

$$\int_0^1 \int_0^{1-x}$$



Change order of Integration

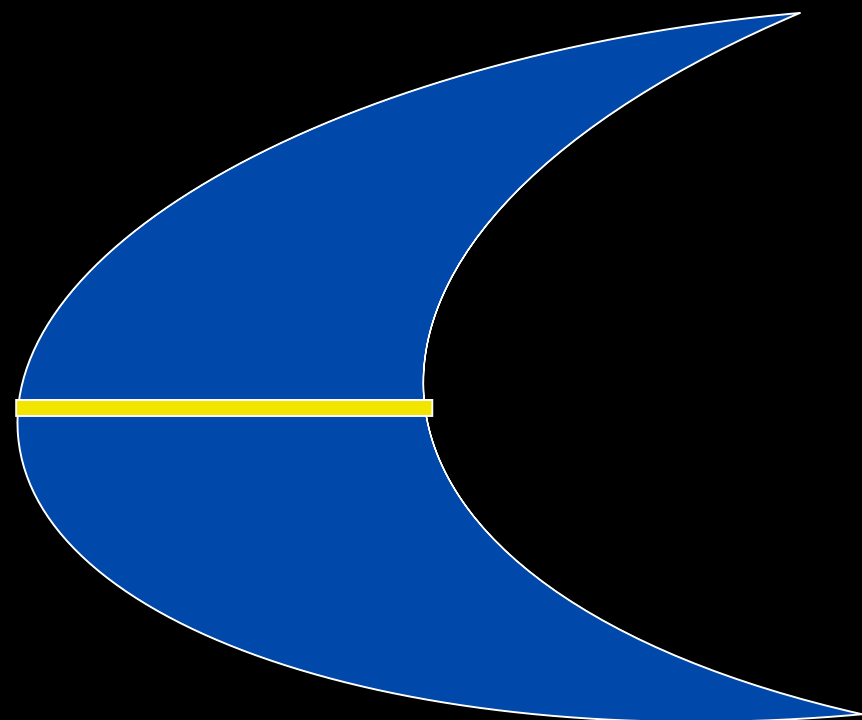


$$\int_0^1 \int_0^{1-x} = \int_0^1 \int_0^{1-y}$$

True or False?

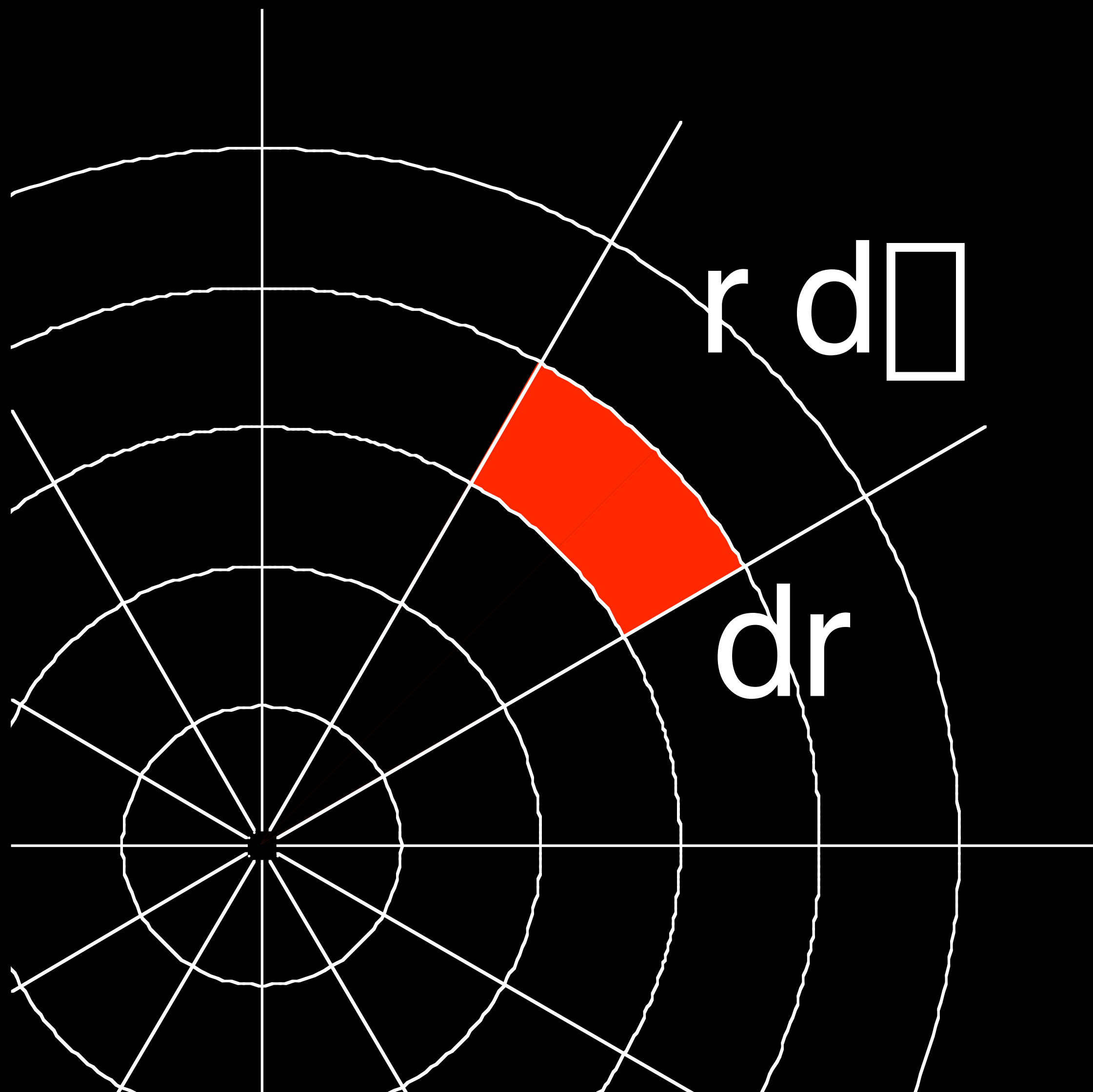
This is a type I region

False



This is a type II region

Polar Integration

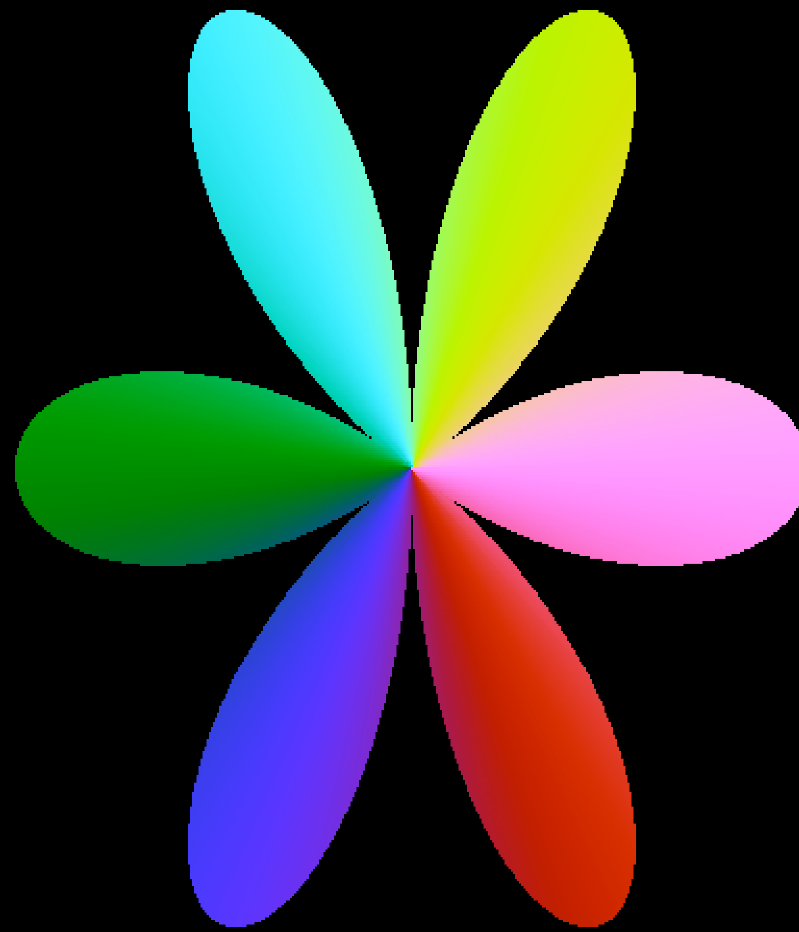


Area of roses

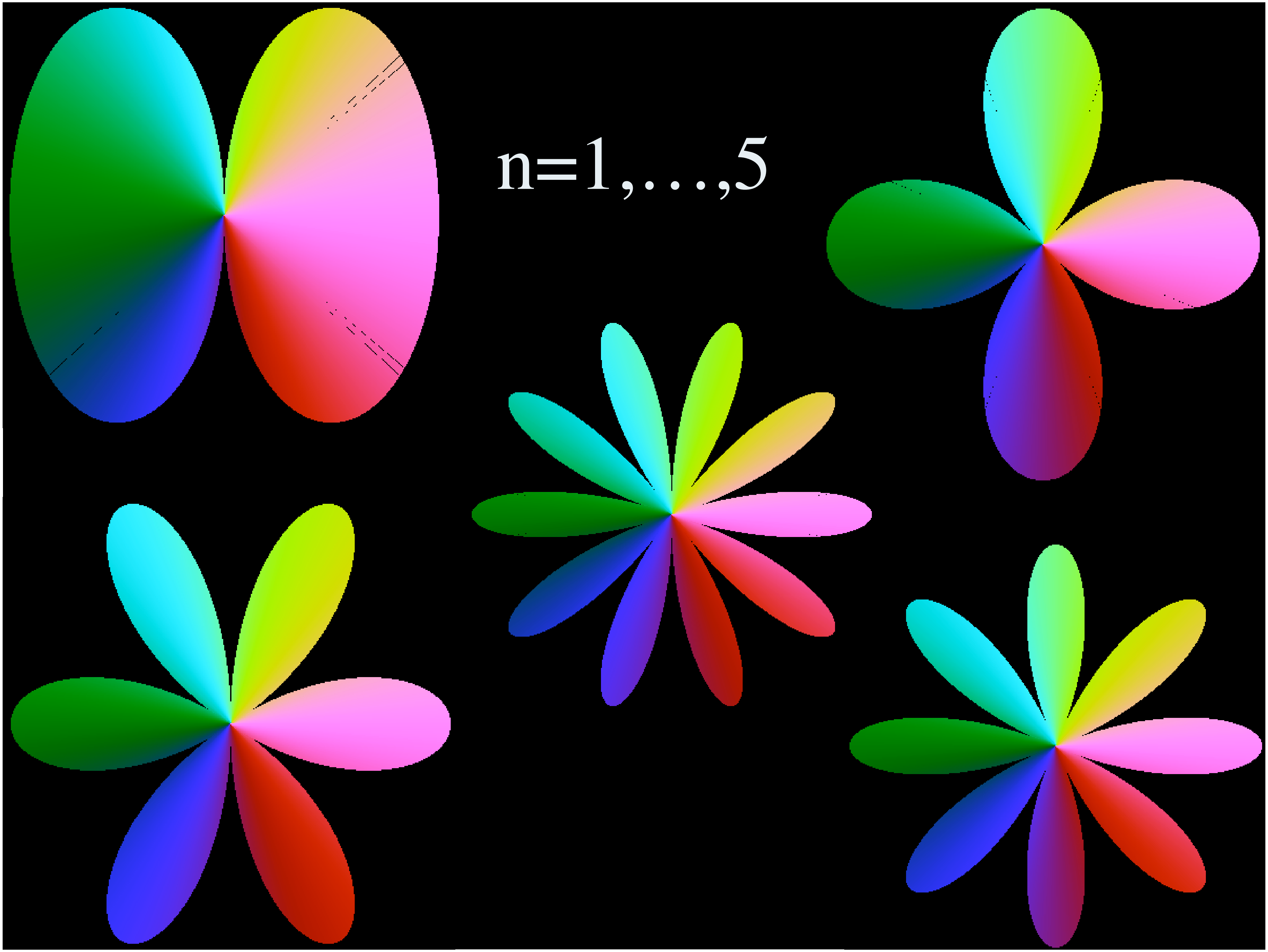
$$r(\theta) = |\cos(n\theta)|$$

Area =

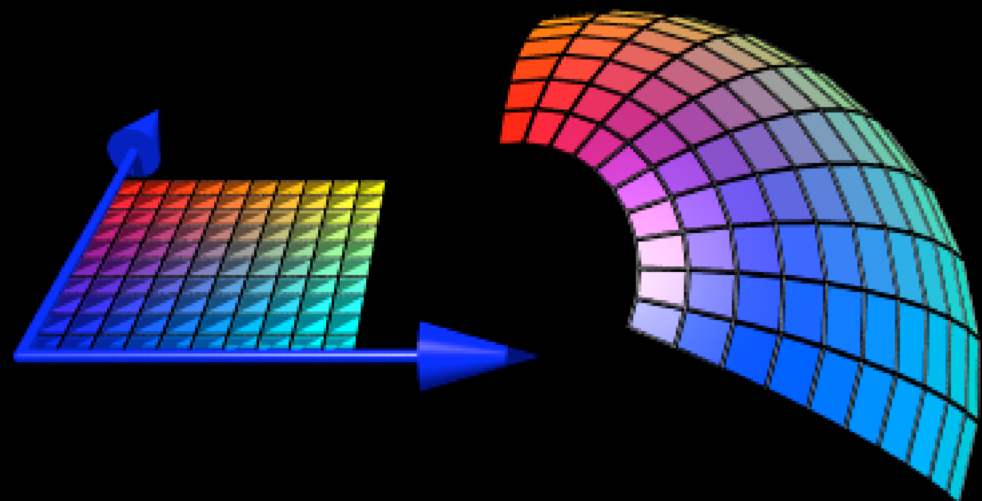
$$\int_0^{2\pi}\int_0^{r(\theta)} r \, dr \, d\theta$$



$$= \int_0^{2\pi} \frac{\cos^2(n\theta)}{2} \, d\theta = \frac{\pi}{2}$$


n=1,...,5

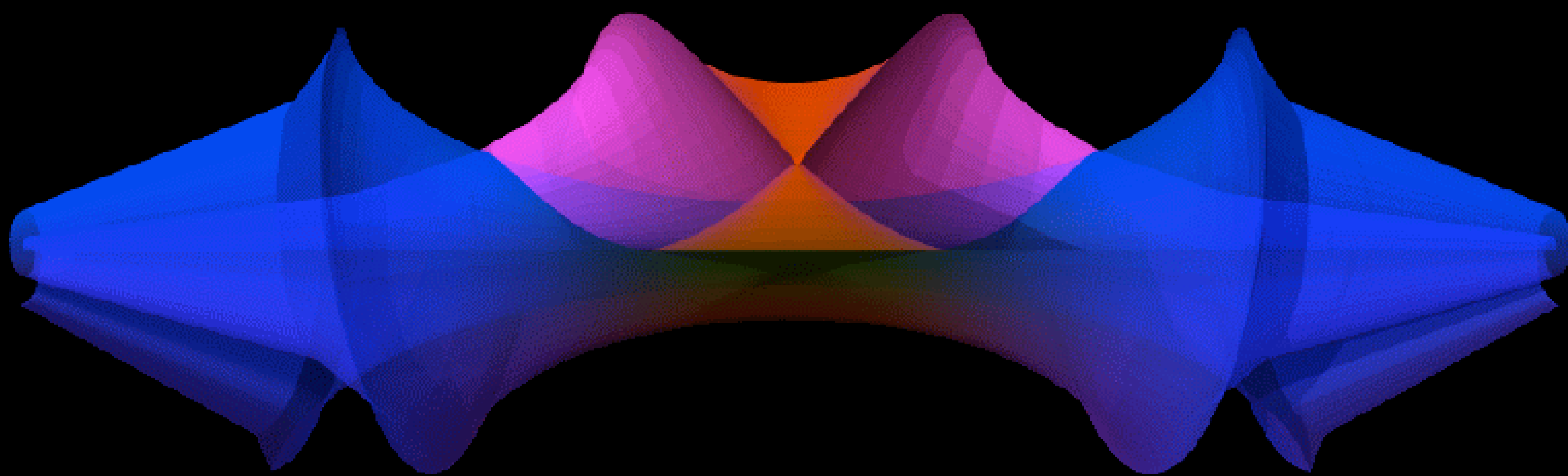
Surface Integrals



$du\,dv$

$|r_u \times r_v|\, du\,dv$

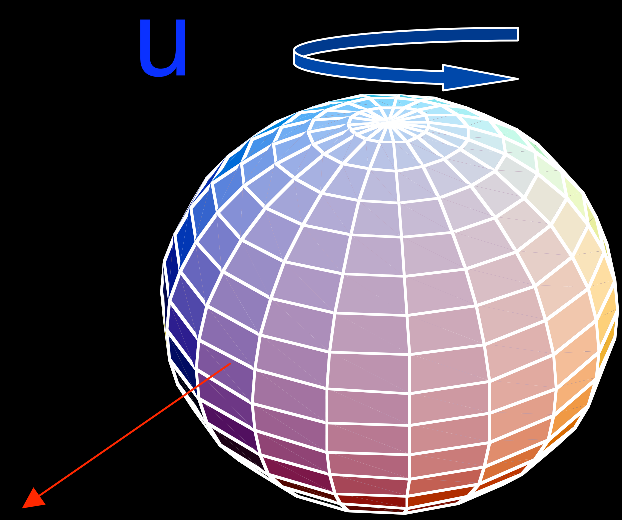
Surface Integrals



$$r(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$r(R) = S$$
$$\text{Area}(S) = \int \int_R |r_u \times r_v| \, du dv$$

Example:

Surface area of the sphere



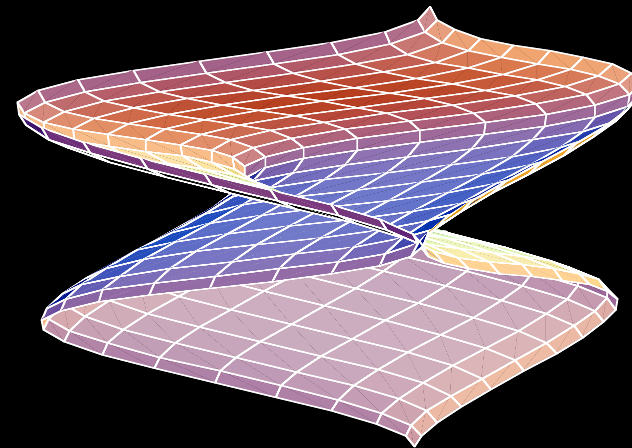
$$r(u,v) = \rho(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)).$$

$$R = [0, 2\pi] \times [0, \pi].$$

$$|r_u(u,v) \times r_v(u,v)| = \rho^2 \sin(\phi)$$

$$\int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\, d\phi d\theta = \boxed{4\pi\rho^2}$$

Example



$r(u,v) = (\cos(u), \cos(v), u+v)$

$R = [0, 2\pi] \times [-\pi, \pi].$

$|r_u(u,v) \times r_v(u,v)|$

$= |(-\sin(u), 0, 1) \times (0, -\sin(v), 1)|$

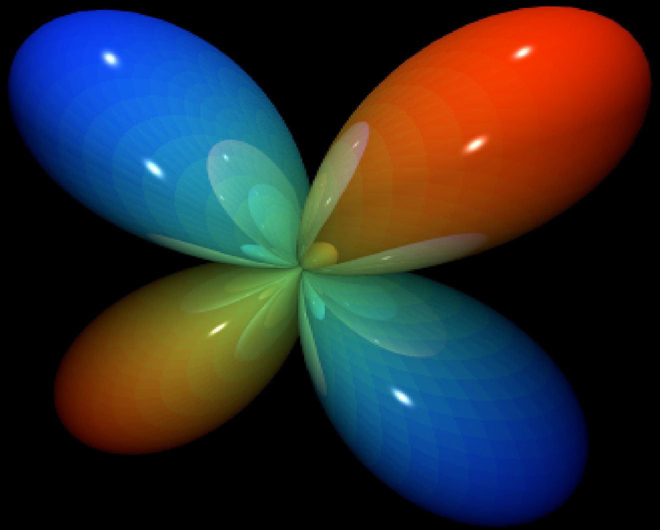
$= |(\sin(v), \sin(u) \sin(u) \sin(v))|$

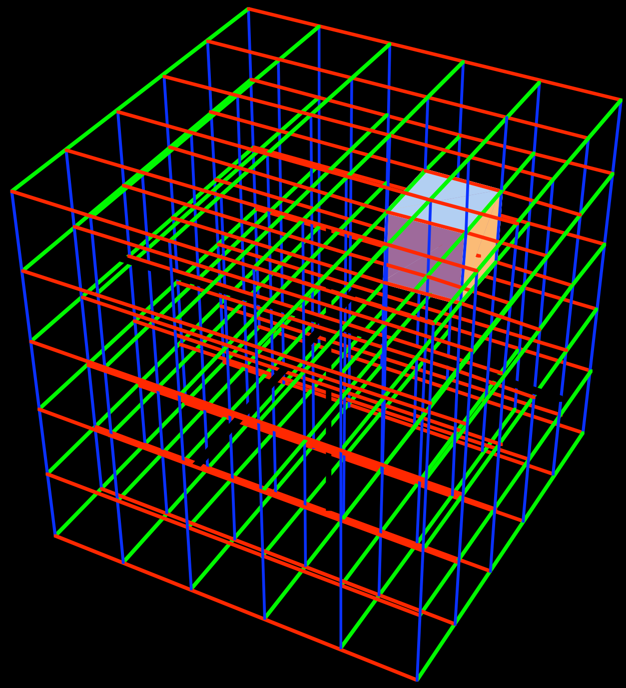
$= \sqrt{\sin^2(u) + \sin^2(v) + \sin^2(u)\sin^2(v)}$

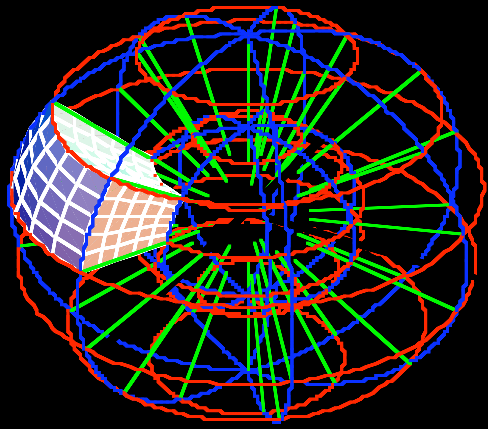
$\int_{-\pi}^{\pi} \int_0^{2\pi} \sqrt{\sin^2(u) + \sin^2(v) + \sin^2(u)\sin^2(v)}\ dudv$

$\boxed{41.67}$ (numerical)

Triple Integrals







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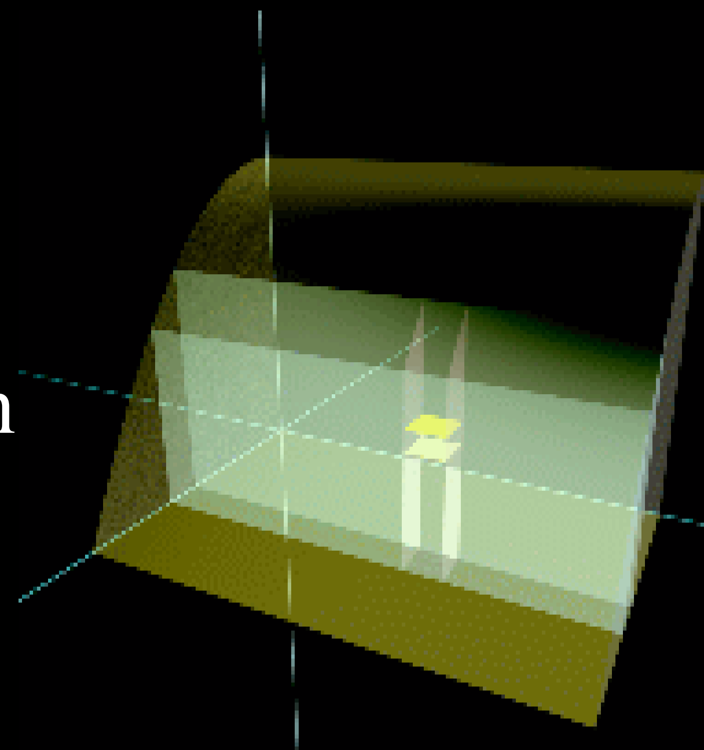
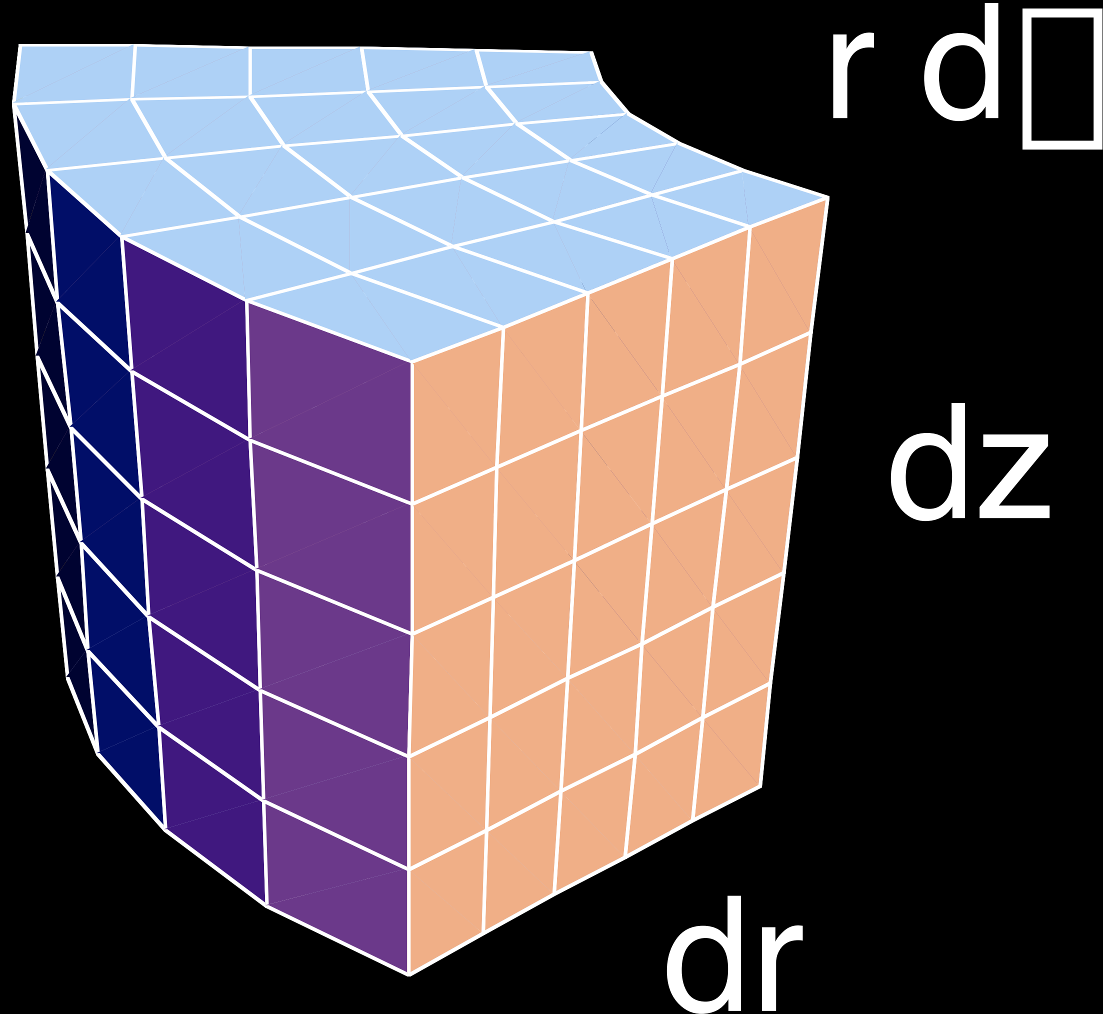
Problem 3

Triple Integrals

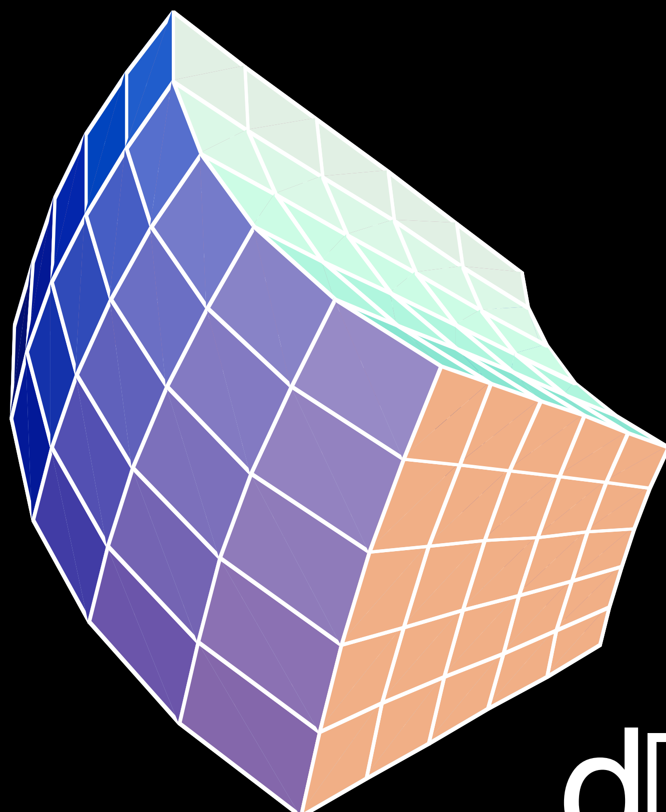
- Type I,II,III regions
- Cylindrical coordinates
- Spherical coordinates
- Change order of integration
- Volume computations



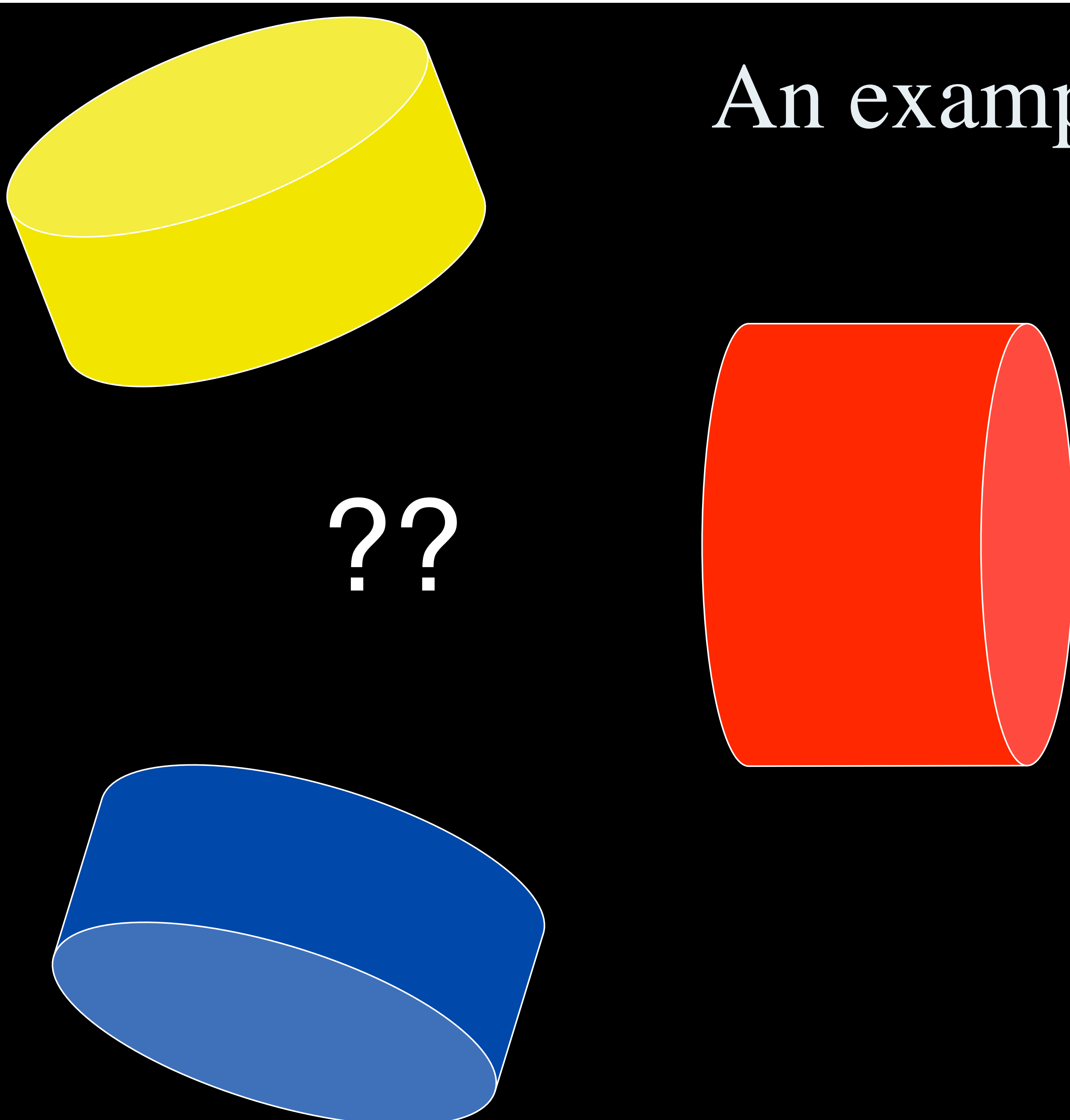
Cylindrical Coordinates



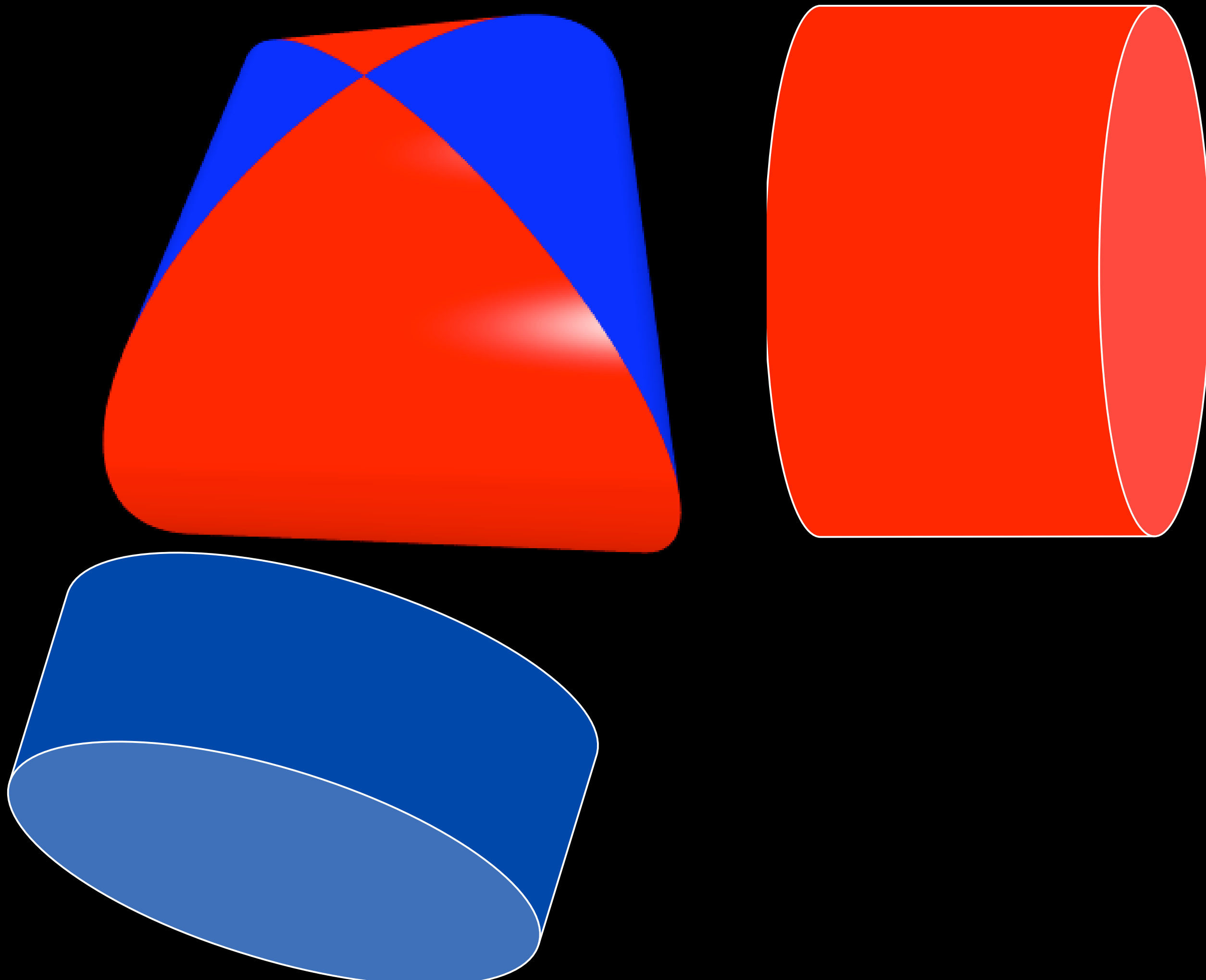
Spherical Coordinates



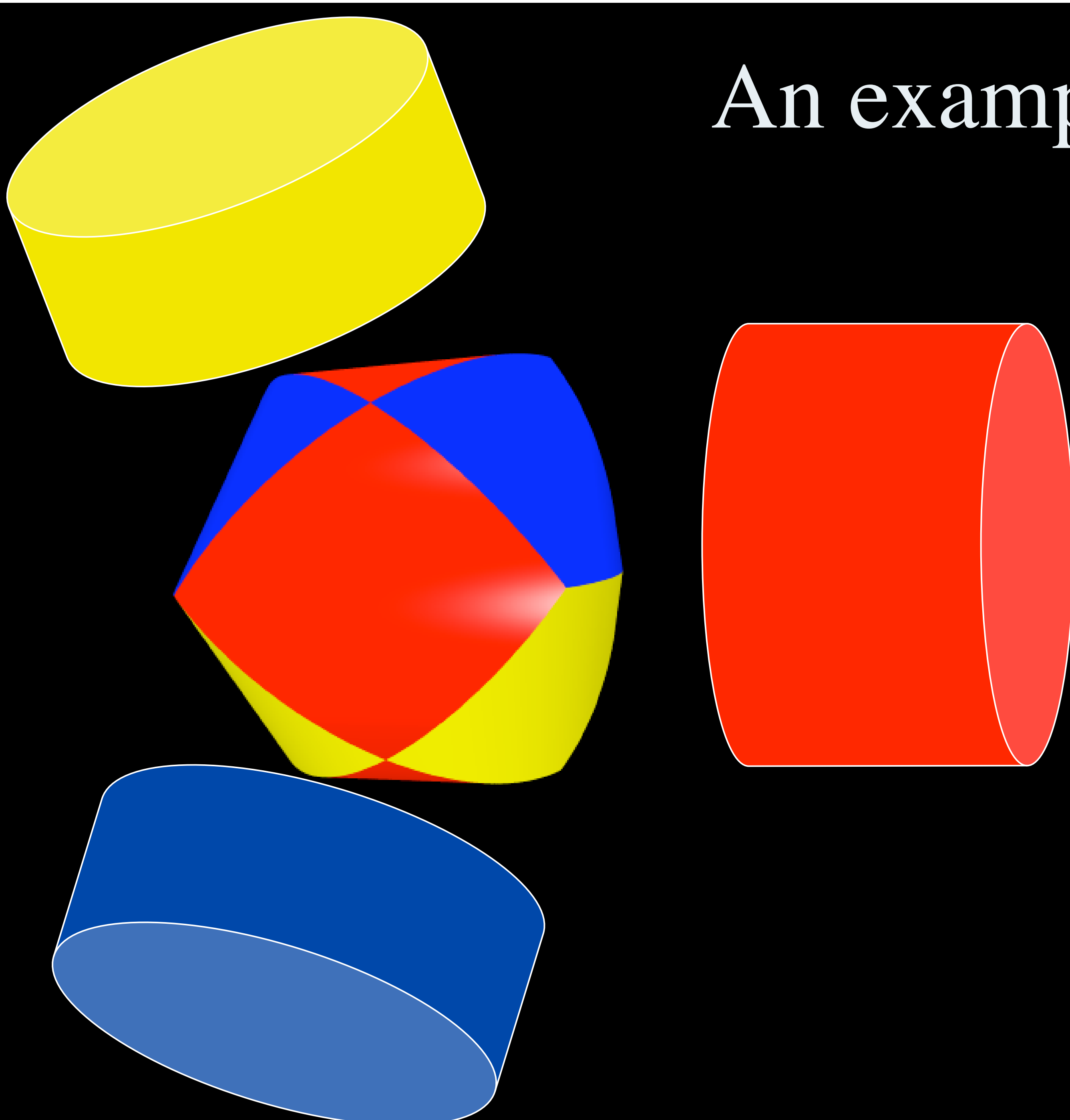
An example



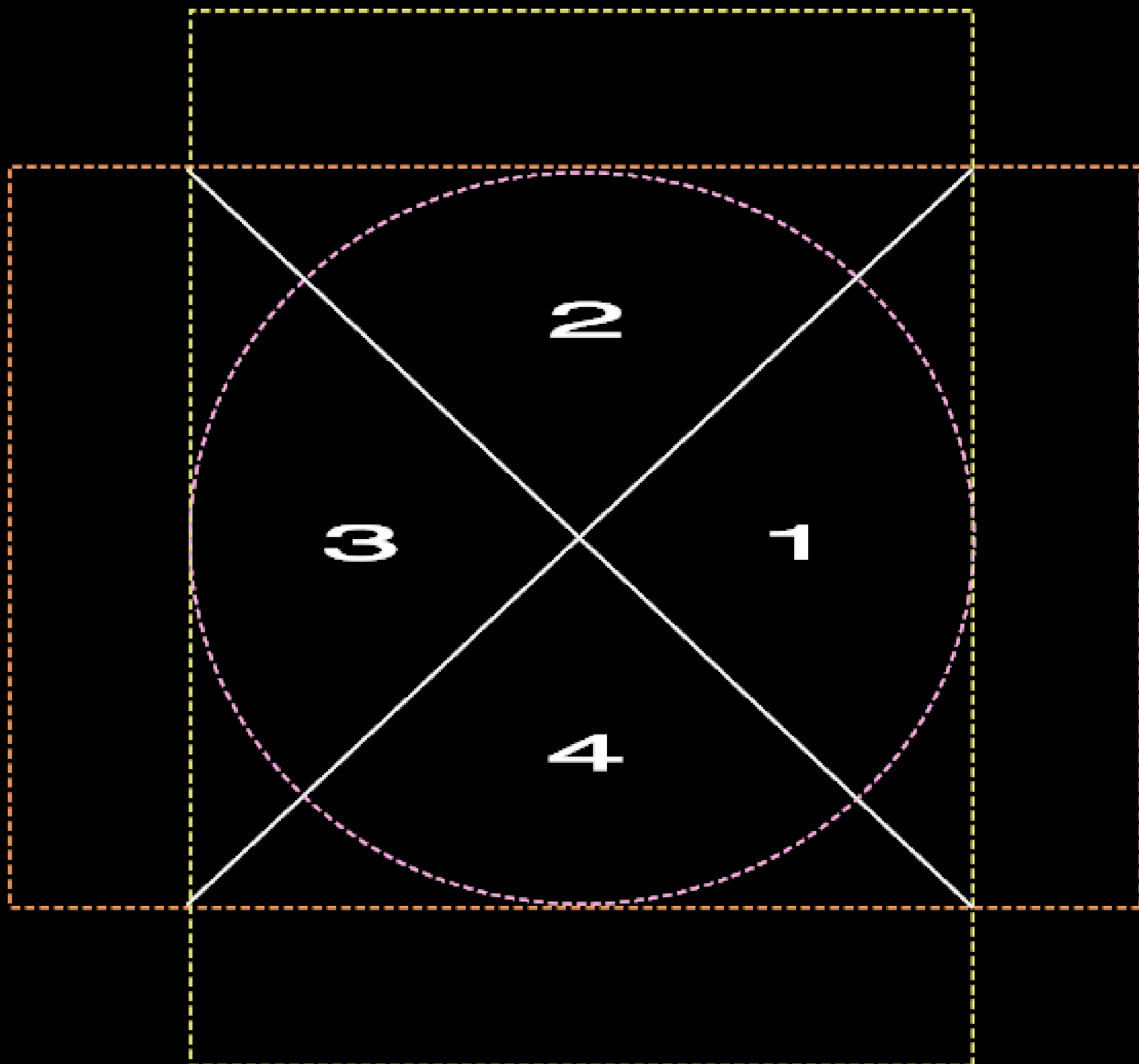
An example



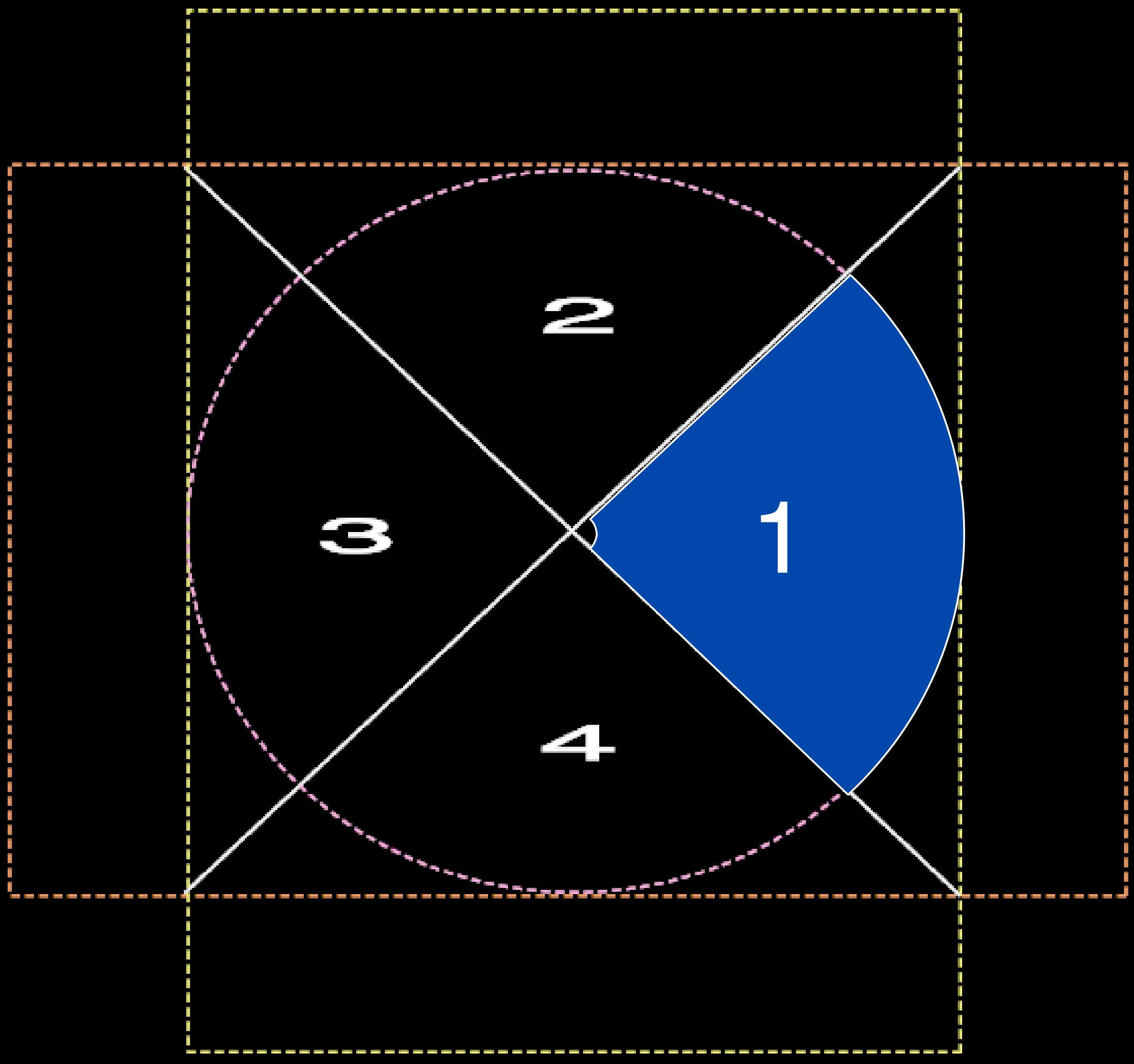
An example



Key Picture



Key Picture



The Integral

$$8\int_{-\pi/4}^{\pi/4}\int_0^1 \sqrt{1-r^2\sin^2(t)}r\,drdt = -\frac{16}{3}+8\sqrt{2}$$


"That's all Folks!"
The end