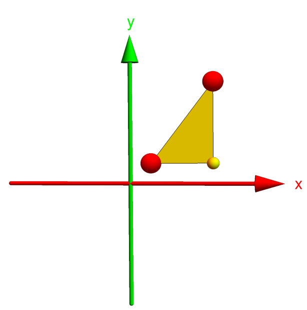
Unit 1: Geometry and Distance

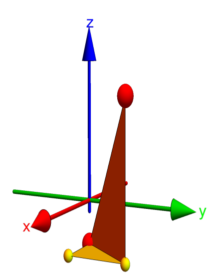
A point in the **plane** $\mathbb{R}^2$ has two **coordinates** $P = (x, y)$ like $P = (2, -3)$. A point in space is determined by three coordinates $P = (x, y, z)$ like $P = (1, 2, 4)$. The plane is divided into 4 **quadrants**, and space is divided into 8 **octants**. The point $P = (1, 2, 4)$ is in the **first octant**. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

1 Describe the location of the points $P = (-1, -2, -3), Q = (0, 0, -5), R = (1, 2, -3)$ in words. **Possible Answer:** $P = (1, 2, 3)$ is in the negative octant of space, where all coordinates are negative. The point $R = (0, 0, -5)$ is on the negative z-axis. The point $S = (1, 2, -3)$ is below the xy-plane. When projected onto the xy-plane it is in the first quadrant.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 9)$. **Answer.** The midpoint is obtained by taking the average of the coordinates: $M = (P + Q)/2 = (-1, 3, 7)$.

The **Euclidean distance** between $P = (x, y)$ and $Q = (a, b)$ is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2}$. The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.





The Euclidean distance is **motivated** by the **Pythagorean theorem**. [1]

3 Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

A **circle** of radius r centered at $P = (a, b)$ consists of all points in the plane with constant distance r from P. The equation is $(x - a)^2 + (y - b)^2 = r^2$.

A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have constant distance ρ from P. The equation of a sphere is $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

[1] It appears in an appendix to "Geometry" of "Discours de la méthode" from 1637, **René Descartes** (1596-1650). More about Descartes in Aczel's book "Descartes Secret Notebook".

4 Is the point $(3,4,5)$ outside or inside the sphere $(x-2)^2+(y-6)^2+(z-2)^2=25$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1+4+9}$ which is smaller than 5, the radius of the sphere. The point is inside.

> The **completion of the square** of an equation $x^2+bx+c=0$ is the idea to add $(b/2)^2-c$ on both sides to get $(x+b/2)^2=(b/2)^2-c$. Solving for x gives the solution $x=-b/2\pm\sqrt{(b/2)^2-c}$.

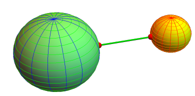
[2]

5 Solve $2x^2-10x+12=0$. **Answer.** The equation is equivalent to $x^2+5x=-6$. Adding $(5/2)^2$ on both sides gives $(x+5/2)^2=1/4$ so that $x=2$ or $x=3$.

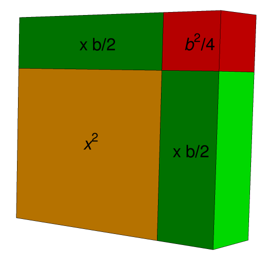
6 Find the center of the sphere $x^2+5x+y^2-2y+z^2=-1$. **Answer:** Complete the square to get $(x+5/2)^2-25/4+(y-1)^2-1+z^2=-1$ or $(x-5/2)^2+(y-1)^2+z^2=(5/2)^2$. We see a sphere **center** $(5/2,1,0)$ and **radius** $5/2$.

Al-Khwarizai

Rene Descartes



Distance between spheres



Completion of Squares

7 Find the set of points $P=(x,y,z)$ in space which satisfy $x^2+y^2=9$. **Answer:** This is a cylinder of radius 3 around the z-axes parallel to the y axis. $\sqrt{x^2+y^2}$ is the distance to the z-axes.

8 What is $x^2+y^2=z^2$. **Answer**: this is the set of points for which the distance to the z axes is equal to the distance to the xy-plane. It must be a cone.

9 Find the distances of $P=(12,5,3)$ to the xy-plane. **Answer:** 3. Find the distance of $P=(12,5,0)$ to z axes. **Answer:** 13.

10 Describe $x^2+2x+y^2-16y+z^2+10z+54=0$. **Answer:** Complete the square to get a sphere $(x+2)^2+(y-8)^2+(z+5)^2=36$ with center $(-2,8,-5)$ and radius 6.

11 Describe the set $xz=x$. **Answer:** We either must have $x=0$ or $z=1$. The set is a union of two coordinate planes.

12 Find an equation for the set of points which have the same distance to $(1,1,1)$ and $(0,0,0)$. **Answer:** $(x-1)^2+(y-1)^2+(z-1)^2=x^2+y^2+z^2$ gives $-2x+1-2y+1-2z+1=0$ or $2x+2y+2z=3$. This is the equation of a plane.

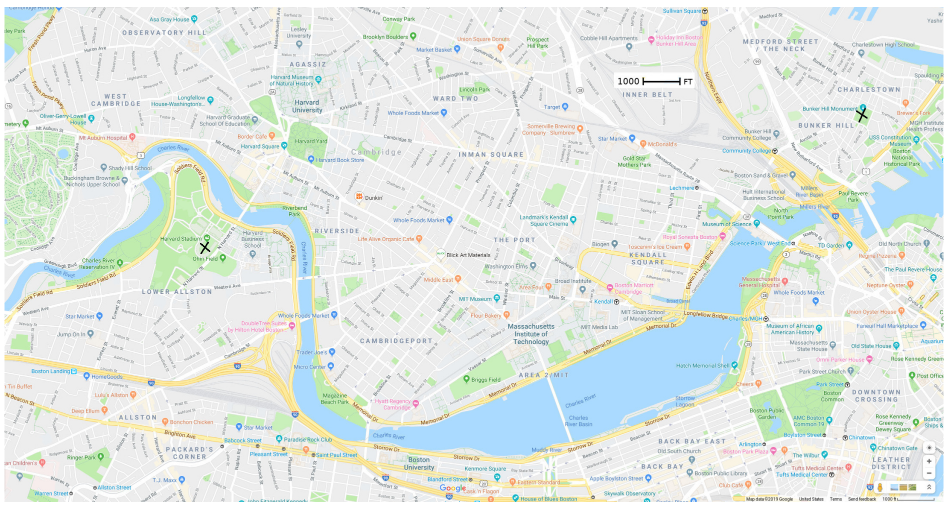
13 Find the distance between the spheres $x^2+(y-12)^2+z^2=1$ and $(x-3)^2+y^2+(z-4)^2=9$. **Answer:**The distance between the centers is $\sqrt{3^2+4^2+12^2}=13$. The distance between the spheres is $13-3-1=9$.

[2]Due to **Al-Khwarizmi** (780-850) in *"Compendium on Calculation by Completion and Reduction"* The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Source book, Ed Victor Katz, contains translations of some of this work.

Unit 1: Worksheet

1 Which points in the plane satisfy the equation $x^2 - 2x + y^2 = 3$?

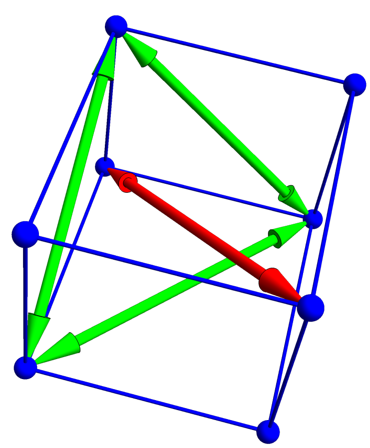
2 Estimate the distance from the Harvard Stadium to the Bunker Hill Monument by measuring horizontal and vertical distances.



3 Can you describe the set $x^2 + (y-1)^2 = 9$ in space?

4 What is the set $x^2 = y^2$ in space?

5 An **Euler brick** is a cuboid of dimensions a,b,c such that $a^2 + b^2$ and $a^2 + c^2$ and $b^2 + c^2$ are all squares. Verify $(a, b, c) = (240, 117, 44)$ leads to an Euler brick. If also the space diagonal is an integer, an Euler brick is called a **perfect cuboid**. It is an open mathematical problem, whether a perfect cuboid exists.



[1]

[1]Hints: 1) Complete the square to see that this is a circle in the plane. 2) About 18'000 feet. The x difference is about 17'500 feet and the y difference is about 4'000 feet. 3) This is a cylinder in space. 4) This is a union of two planes $x = y$ and $x = -y$.

2: Vectors and Dot product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{PQ} = \vec{v} = [x-a, y-b-z-c]$ pointing from P to Q. The real numbers v_1, v_2, v_3 in $\vec{v} = [v_1, v_2, v_3]$ are the **components** of $\vec{v}$.

Similar definitions hold in two dimensions, where vectors have two components. Vectors can be drawn **everywhere** in space. Two vectors with the same components are considered **equal**. [1]

The **addition** of two vectors is $\vec{u}+\vec{v} = [u_1, u_2, u_3]+[v_1, v_2, v_3] = [u_1+v_1, u_2+v_2, u_3+v_3]$. The **scalar multiple** $\lambda\vec{u} = \lambda[u_1, u_2, u_3] = [\lambda u_1, \lambda u_2, \lambda u_3]$. The difference $\vec{u} - \vec{v}$ can be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The addition and scalar multiplication of vectors satisfy the laws you know from **arithmetic**. **commutativity** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, **associativity** $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ and $r*(s*\vec{v}) = (r*s)*\vec{v}$ as well as **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$ and $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$, where $*$ is scalar multiplication.

The **length** or **magnitude** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. A synonym is **direction**. Nonzero vectors have length and magnitude.

1 $|[3, 4]| = 5$ and $|[3, 4, 12]| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $[3/5, 4/5]$ and $[3/13, 4/13, 12/13]$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

The dot product determines distance and distance determines the dot product.

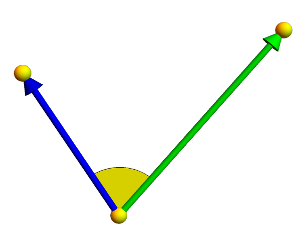
Proof: Using the dot product one can express the length of $\vec{v}$ as $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. On the other hand, $(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2(\vec{v} \cdot \vec{w})$ allows to solve for $\vec{v} \cdot \vec{w}$:

$$\vec{v} \cdot \vec{w} = (|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2 .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. We only need to show it in the case $|\vec{w}| = 1$. Define $a = \vec{v} \cdot \vec{w}$ and estimate $0 \leq (\vec{v}-a\vec{w})\cdot(\vec{v}-a\vec{w})$ to get $0 \leq (\vec{v}-(\vec{v}\cdot\vec{w})\vec{w})\cdot(\vec{v}-(\vec{v}\cdot\vec{w})\vec{w}) = |\vec{v}|^2+(\vec{v}\cdot\vec{w})^2-2(\vec{v}\cdot\vec{w})^2 = |\vec{v}|^2-(\vec{v}\cdot\vec{w})^2$ which means $(\vec{v} \cdot \vec{w})^2 \leq |\vec{v}|^2$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$.



[1] We cover 2400 years of math from Pythagoras (500 BC), Al Kashi (1400), Cauchy (1800) to Hamilton (1850).

Al Kashi's theorem: A triangle ABC with side lengths a, b, c and angle α opposite to c satisfies $a^2 + b^2 = c^2 + 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

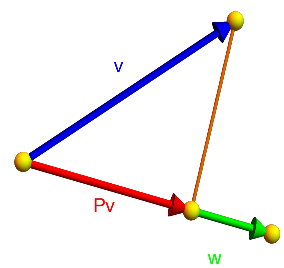
Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2+\vec{v}^2+2\vec{u}\cdot\vec{v} \leq \vec{u}^2+\vec{v}^2+2|\vec{u}\cdot\vec{v}| \leq \vec{u}^2+\vec{v}^2+2|\vec{u}|\cdot|\vec{v}| = (|\vec{u}|+|\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$.

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. [2]

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$.



2 Find the projection of $\vec{v} = [0, -1, 1]$ onto $\vec{w} = [1, -1, 0]$. **Answer:** $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]$.

3 A wind force $\vec{F} = [2, 3, 1]$ is applied to a car which drives in the direction of the vector $\vec{w} = [1, 1, 0]$. Find the projection of $\vec{F}$ onto $\vec{w}$, the force which accelerates or slows down the car. **Answer**: $\vec{w}(\vec{F} \cdot \vec{w}/|\vec{w}|^2) = [5/2, 5/2, 0]$.

4 How can we visualize the dot product? **Answer**: the absolute value of the dot product is the length of the projection. Positive dot product means $\vec{v}$ and $\vec{w}$ form an acute angle, negative if that angle is obtuse.

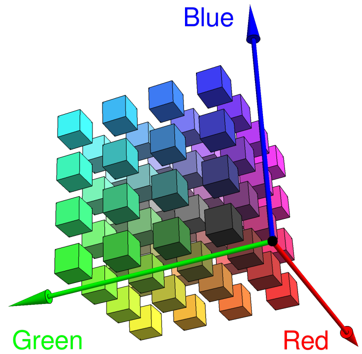
5 Given $\vec{v} = [2, 1, 2]$ and $\vec{w} = [3, 4, 0]$. Find a vector which is in the plane defined by $\vec{v}$ and $\vec{w}$ and which bisects the angle between these two vectors. **Answer**. Normalize the two vectors to make them unit vectors then add them to get $[13, 17, 10]/15$.

6 Given two vectors $\vec{v}, \vec{w}$ which are perpendicular. Under which condition is $\vec{v}+\vec{w}$ perpendicular to $\vec{v} - \vec{w}$? **Answer**: Find the dot product of $\vec{v} + \vec{w}$ with $\vec{v} - \vec{w}$ and set it zero.

7 Is the angle between $[1, 2, 3]$ and $[-15, 2, 4]$ acute or obtuse? **Answer**: the dot product is 1. Ah! Cute!

[2] We have just proven Pythagoras and Al Kashi. Distance and angle were defined, not deduced.

Unit 2: Dot product Worksheet



Colors are encoded by vectors $\vec{v} = [r, g, b]$, where the **red**, **green** and **blue** components are all numbers in the interval $[0, 1]$. Examples are:

$[0, 0, 0]$	black	$[0, 0, 1]$	blue
$[1, 1, 1]$	white	$[1, 1, 0]$	yellow
$[1/2, 1/2, 1/2]$	gray	$[1, 0, 1]$	magenta
$[1, 0, 0]$	red	$[0, 1, 1]$	cyan
$[0, 1, 0]$	green	$[1, 1/2, 0]$	orange
$[0, 1, 1/2]$	spring green	$[1, 1, 1/2]$	khaki
$[1, 1/2, 1/2]$	pink	$[1/2, 1/4, 0]$	brown

1 Determine the angle between the colors magenta and cyan.

2 Find a color which is both orthogonal to orange and yellow.

3 What does the scaling $\vec{v} \mapsto \vec{v}/2$ do, if $\vec{v}$ represents a color?

[1]

[1] Hints: 1) We have to find the angle between $[1, 0, 1]$ and $[1, 1, 0]$. This is 60 degrees, 2) Blue, 3) We make the color darker

3: Cross product

The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]$ and $\vec{w} = [w_1, w_2, w_3]$ in space is defined as the vector

$$\vec{v} \times \vec{w} = [v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1] \, .$$

To remember it, we write the product as a "determinant":

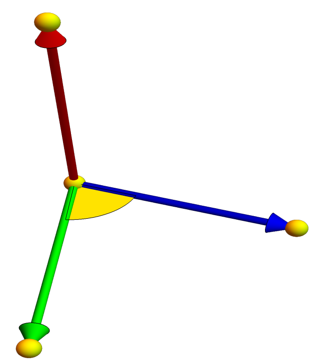
$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$. [1]

1 The cross product of $[1, 2, 3]$ and $[4, 5, 1]$ is the vector $[-13, 11, -3]$.

The cross product $\vec{v} \times \vec{w}$ is anti-commutative. The resulting vector is orthogonal to both $\vec{v}$ and $\vec{w}$.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.



The **sin** formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof: We verify the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda \vec{w}$ for some real λ.

Proof. This follows immediately from the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel even so sometimes one calls this anti-parallel.

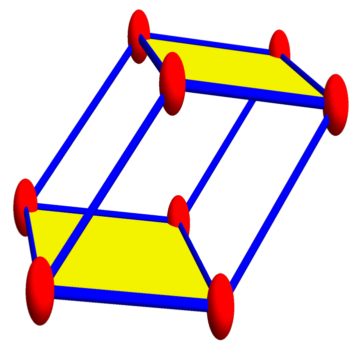
[1] It was Hamilton who found in 1843 a multiplication $*$ of 4 vectors. It contains both dot and cross product because $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-vw, v \times w)$.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

Proof. Twice the area of the triangle is $ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta)$ Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

2 If $\vec{v} = [a, 0, 0]$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]$, then $\vec{v} \times \vec{w} = [0, 0, ab\sin(\alpha)]$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$ and the **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$.



The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is indeed the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand.

3 **Problem:** Find the volume of a **cuboid** of width a length b and height c. **Answer**. The cuboid is a parallelepiped spanned by $[a, 0, 0]$ $[0, b, 0]$ and $[0, 0, c]$. The triple scalar product is abc.

4 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that it is spanned by the vectors $\vec{u} = [1, 2, 1]$, $\vec{v} = [3, 2, 1]$, and $\vec{w} = [0, 3, 1]$. We get $\vec{v} \times \vec{w} = [-1, -3, 9]$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

5 **Problem:** find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$. To do so find a vector $\vec{n} = [a, b, c]$ normal to the and noting $(\vec{x} - \vec{OP}) \cdot \vec{n} = 0$. **Answer:** A normal vector $\vec{n} = [1, -2, 2] = [a, b, c]$ of the plane $ax + by + cz = d$ is obtained as the cross product of $\vec{PQ}$ and $\vec{RQ}$ With $d = \vec{n} \cdot \vec{OP} = [1, -2, 2] \cdot [1, 2, 3] = 3$, we get the equation $x - 2y + 2z = 3$.

The cross product appears in physics, like for the angular momentum, the Lorentz force or the Coriolis force. We will however mainly use the cross product for constructions like to get the equation of a plane through 3 points A, B, C.

3: Lines and Planes

A point $P = (x_0, y_0, z_0)$ and a vector $\vec{v} = [a, b, c]$ define the **line**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} .$$

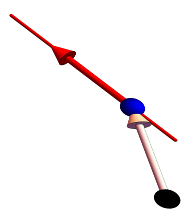
It is called the **parameterization** of the line.

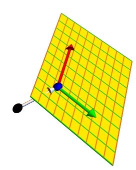
Every vector contained inside the line is parallel to $\vec{v}$. We think about the parameter t as "time" and about $\vec{v}$ as the **velocity**. For $t = 0$, we are at P identified with $\vec{OP}$.

Given two points like $(2, 3, 4)$ with $(3, 3, 5)$, the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

connects the points $(2, 3, 4)$ with $(3, 3, 5)$. If t is restricted to $[0, 1]$ we get the **line segment** defined by the two points.





1 **Problem.** Parametrize the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$.
Solution. with $\vec{v} = \vec{PQ} = [1, 3, 4]$ we get get the line

$$[x, y, z] = [1, 1, 2] + t[1, 3, 4]$$

which is $\vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t]$. Since $[x, y, z] = [1, 1, 2] + t[1, 3, 4]$ consists of three equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ we can solve each for t to get $t = (x - 1)/2 = (y - 1)/3 = (z - 2)/4$.

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit if one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

2 Find the symmetric equations for the line through the two points $P = (0,1,1)$ and $Q = (2,3,4)$ **Solution.** first first form the parametric equations $[x,y,z] = [0,1,1] + t[2,2,3]$ or $x = 2t, y = 1+2t, z = 1+3t$ and solve for t to get $x/2 = (y-1)/2 = (z-1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a,b,c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\vec{r}(t) = \vec{OP} + t\vec{v} + s\vec{w}$, where t,s are real numbers.
This is called the **parametric description** of a plane.

A point $P = (x_0, y_0, z_0)$ and two vectors $\vec{v}, \vec{w}$ defines the **plane**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + s \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} .$$

It is called the **parameterization** of the plane.

2

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} .$$

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector

$$\vec{n} = \vec{v} \times \vec{w}$$

is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q-P)\cdot\vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x,y,z)$, and $\vec{n} = [a,b,c]$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$, where d is a constant obtained by plugging in a point. We have just shown

The equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$

$$ax + by + cz = d ,$$

where $[a,b,c] = \vec{v} \times \vec{w}$ and d is obtained by plugging in $\vec{x}_0$.

3 **Problem:** Find the equation of a plane which contains the three points $P = (-1,-1,1), Q = (0,1,1), R = (1,1,3)$.
Answer: The plane contains the two vectors $\vec{v} = [1,2,0]$ and $\vec{w} = [2,2,2]$. We have $\vec{n} = [4,-2,-2]$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

4 **Problem:** Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. **Answer:** find the angle between $\vec{n} = [1,1,0]$ and $\vec{m} = [1,1,1]$. It is $\arccos(2/\sqrt{6})$.

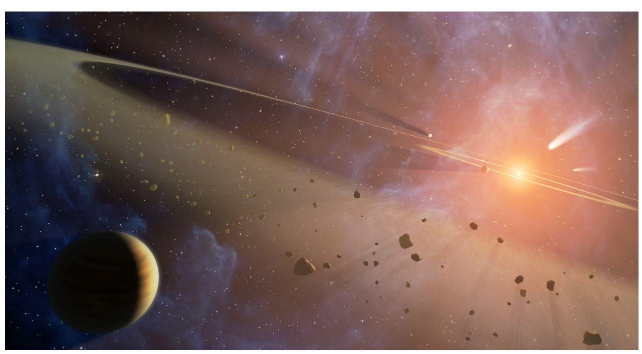
Unit 3: Worksheet

1 What is $\left(\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \times \begin{bmatrix}1\\0\\0\end{bmatrix}$?

2 The star Proxima Centauri is a bit more than 4 light years away from us. The brightest star we see in the night sky is **Sirius**. It is 8.6 light years away. The star **Epsilon Eridani** is the closest star for which one has confirmed a planet. It is 10.5 light years away from us. Assume the earth is at $A = (1, 0, 0)$, Proxima Centauri at $B = (4, 3, 0)$ and Sirius at $C = (7, 4, 3)$. Find the plane $ax + by + cz = d$ containing A, B, C.

3 What is the distance of Epsilon Eridani, a star at $D = (8, -5, -6)$ to that plane?

4 Parametrize the line through D perpendicular to the plane ABC.



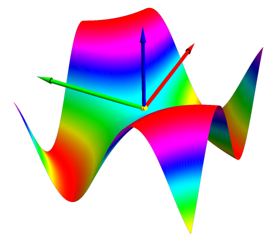
Source: Caltech. Artist rendering of asteroid belts in Epsilon Eridani.

[1]

[1]Hints:
1) $[0, 1, 0]$.
2) Build $\vec{AB} = [3, 3, 0]$ and $\vec{AC} = [6, 4, 3]$. The cross product gives $\vec{n} = [-9, -9, -6] = [a, b, c]$ perpendicular to them. The equation of the plane is $ax + by + cz = -9x - 9y - 6z = d$. To get d plug in a point like $(1, 0, 0$. We get $d = -9$.] 3) Use $|\vec{n} \cdot \vec{AD}|/|\vec{n}| = 18/\sqrt{198}$. 4) $\vec{r}(t) = [8, -5, -6] + t[-9, -9, -6] = [8 - 9t, -5 - 9t, -6 - 6t]$.

4: Functions of several variables

A **function of two variables** $f(x, y)$ is a rule which assigns to two numbers x, y a third number $f(x, y)$. For example, the function $f(x, y) = x^2y+2x$ assigns to $(3, 2)$ the number $3^2 2 + 6 = 24$. The **domain** D of a function is set of points where f is defined, the range is $\{f(x, y) \mid (x, y) \in D\}$. The **graph** of $f(x, y)$ is the surface $\{(x, y, f(x, y)) \mid (x, y) \in D\}$ in space. Graphs allow to visualize functions.

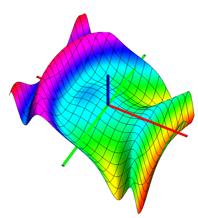


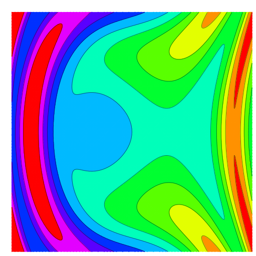
1 The graph of $f(x, y) = \sqrt{1 - (x^2 + y^2)}$ on the domain $D = \{x^2 + y^2 < 1\}$ is a half sphere. The range is the interval $[0, 1]$.

The set $f(x, y) = c = \text{const}$ is called a **contour curve** or **level curve** of f. For example, for $f(x, y) = 4x^2 + 3y^2$, the level curves $f = c$ are ellipses if $c > 0$. The collection of all contour curves $\{f(x, y) = c\}$ is called the **contour map** of f.

2 For $f(x, y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The curve $x^2 - y^2 = 1$ is made of two hyperbola with with their "noses" at the point $(-1, 0)$ and $(1, 0)$. The curve $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0, 1)$ and $(0, -1)$.

3 For complicated functions like $f(x, y) = \sin(x^3 - y^2) - x$, it is difficult to find the contour curves. We can draw the curves with the computer:



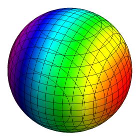


A function of three variables $g(x, y, z)$ assigns to three variables x, y, z a real number $g(x, y, z)$. We can visualize it by **contour surfaces** $g(x, y, z) = c$, where c is constant. It is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

4 For $g(x, y, z) = z - f(x, y)$, the level surface $g = 0$ which is the graph $z = f(x, y)$ of a function of two variables. For example, for $g(x, y, z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2 + y^2$ of the function $f(x, y) = x^2 + y^2$ which is a paraboloid. Most surfaces $g(x, y, z) = c$ are not graphs.

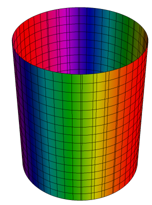
5 If $f(x, y, z)$ is a polynomial and $f(x, x, x)$ is quadratic in x, then $\{f = c\}$ is a **quadric**.

Sphere



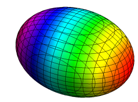
$x^2 + y^2 + z^2 = 1$

Cylinder



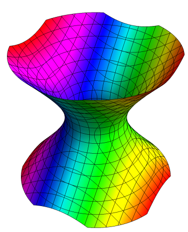
$x^2 + y^2 = 1$

Ellipsoid



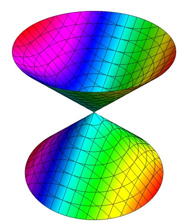
$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

One sheeted Hyperboloid



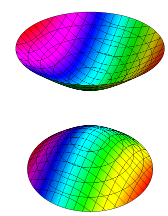
$x^2 + y^2 - z^2 = 1$

Cone



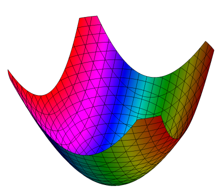
$x^2 + y^2 - z^2 = 0$

Two sheeted Hyperboloid



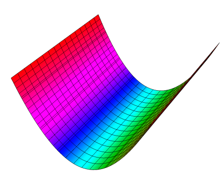
$x^2 + y^2 - z^2 = -1$

Elliptic paraboloid



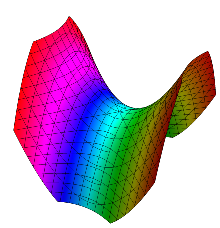
$z = x^2 + y^2$

Cylindrical paraboloid



$z = x^2$

Hyperbolic paraboloid

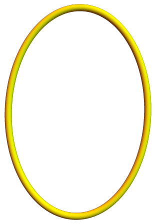


$z = x^2 - y^2$

6 If the function involves only multiplications of variables x, y, z and $x \to f(x, x, x)$ has degree d, then it is called a **degree** d **polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.

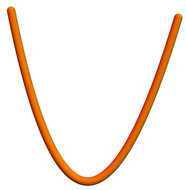
Conic sections

Ellipse



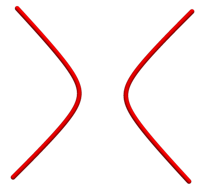
$x^2 + y^2 = 1$

Parabola



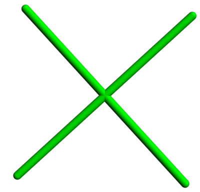
$y = x^2$

Hyperbola



$x^2 - y^2 = 1$

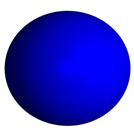
Specials



$x^2 = y^2$

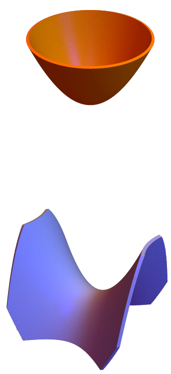
Quadrics

Ellipsoid



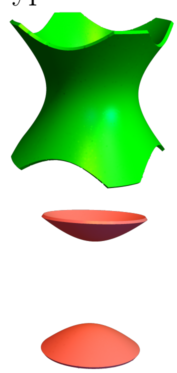
$x^2 + y^2 + z^2 = 1$

Paraboloid



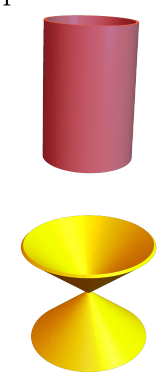
$z = x^2 + y^2$
$z = x^2 - y^2$

Hyperboloid



$x^2 + y^2 - z^2 = 1$
$x^2 + y^2 - z^2 = -1$

Specials



$x^2 + y^2 = 1$
$x^2 + y^2 = z^2$

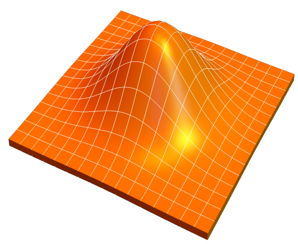
Advise

You will have to know the quadrics in the exam. Here are some pointers:

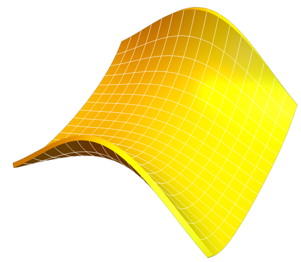
- There is no need to memorize the quadrics. You can derive them: look at the traces (put one of the variables to zero) to see what conic section you get.
- The name usually reveals what the surface is: elliptic paraboloids contain ellipses and parabola, hyperbolic paraboloids contain hyperbola and parabola as traces.
- The paraboloids can be written as graphs $z = f(x, y)$. This is not possible for ellipsoids or hyperboloids.
- The special surfaces are non-generic but useful and important. The cone or cylinder appear a lot in applications.
- Make sure to recognize the surfaces also if the variables are turned, scaled shifted or signs switched. $2x^2 - (y-5)^2 - 4z^2 = -1$ for example is a one-sheeted hyperboloid.

Unit 4: Functions

1 Match the graphs with the functions



A



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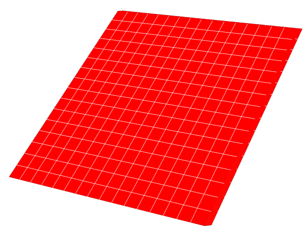

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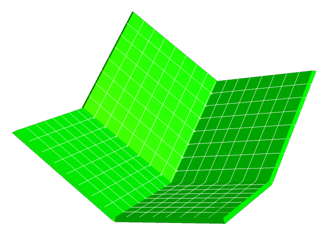

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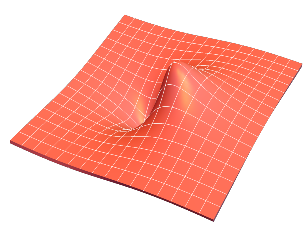
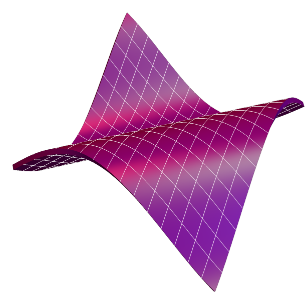


E



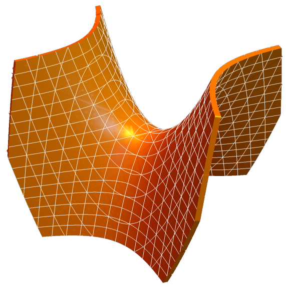
F

- $f(x,y) = x + y$
- $f(x,y) = \sin(x - y)$
- $f(x,y) = e^{-x^2+y^2}$
- $f(x,y) = |x| + |y|$
- $f(x,y) = x/(1 + x^2 + y^2)$
- $f(x,y) = e^{-x^2-y^2}$

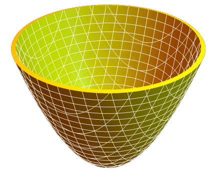
2 Sketch the contour map in each case.

Unit 4: Quadrics worksheet

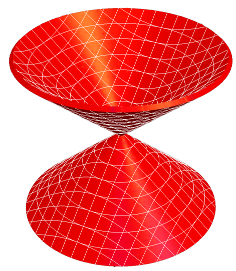
1 What is the name of the following surfaces?



A



B



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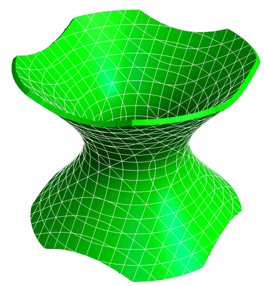
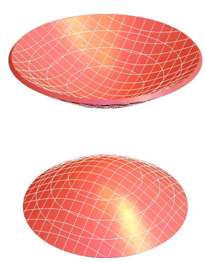


D



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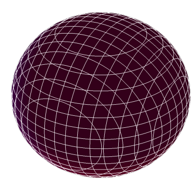


F

2 Match the implicit equations $g(x, y, z) = 0$ with the surfaces.

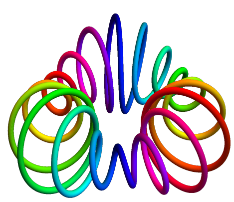
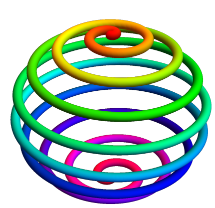
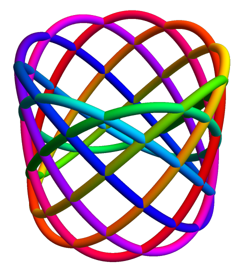
- $x^2 + y^2 + z^2 = 1$
- $x^2 + y^2 - z^2 = 1$
- $x^2 + y^2 - z^2 = -1$
- $x^2 + y^2 + z = 1$
- $x^2 - y^2 + z = 1$
- $x^2 + y^2 = z^2$

5: Curves

We have already seen lines like $\vec{r}(t) = [x(t), y(t), z(t)] = [1+t, 2-3t, 4-3t]$. We can generalize this and replace the entries with general functions $x(t), y(t), z(t)$. Depending on how many coordinates we use, we have either a curve in the plane or a curve in three dimensional space.

A **parametrization** of a planar curve is a map $\vec{r}(t) = [x(t), y(t)]$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. The parametrization of a space curve is $\vec{r}(t) = [x(t), y(t), z(t)]$. The **image** of r is a **parametrized curve** in space.

Here are some pictures of cool curves

We think of the **parameter** t as **time**. For a fixed t, we have a vector $[x(t), y(t), z(t)]$ in space. As t varies, the end point of this vector moves along a curve. The parametrization contains more information about the curve then the curve. It tells also how fast and in which direction we trace the curve.

1 The parametrization $\vec{r}(t) = [2+t, 3+t, 1+t] = [2, 3, 1] + t[1, 1, 1]$ is a **line** in space.

2 The parametrization $\vec{r}(t) = [2+3\cos(t), 4+3\sin(t)]$ is a **circle** of radius 3 centered at $(2, 4)$

3 $\vec{r}(t) = [\cos(3t), \sin(5t)]$ defines a **Lissajous curve** example.

4 If $x(t) = t$, $y(t) = t^2$, the curve $\vec{r}(t) = [t, t^2]$ traces the **graph** of the function $f(t) = t^2$. It is a parabola.

5 With $\vec{r}(t) = [2\cos(t), 5\sin(t)] = [x(t), y(t)]$ describes an **ellipse** $x(t)^2/4 + y(t)^2/25 = 1$.

6 The space curve $\vec{r}(t) = [\cos(t), \sin(t), t]$ traces a **helix**

7 If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$ is the same curve as before but the **parameterization** has changed.

8 With $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$ it is traced in the **opposite direction**.

9 With $\vec{r}(t) = [\cos(t), \sin(t)] + 0.1[\cos(17t, \sin(17t)]$ we have an example of an **epicycle**, where a circle turns on a circle. It was used in the **Ptolemaic geocentric system** which predated the Copernican system still using circular orbits and then the modern Keplerian system, where planets move on ellipses and which can be derived from Newton's laws.

If $\vec{r}(t) = [x(t), y(t), z(t)]$ is a curve, then $\vec{r}\,'(t) = [x'(t), y'(t), z'(t)] = [\dot{x}, \dot{y}, \dot{z}]$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**. Then come **snap**, **crackle**, **pop** and (if you like) the **Harvard**.

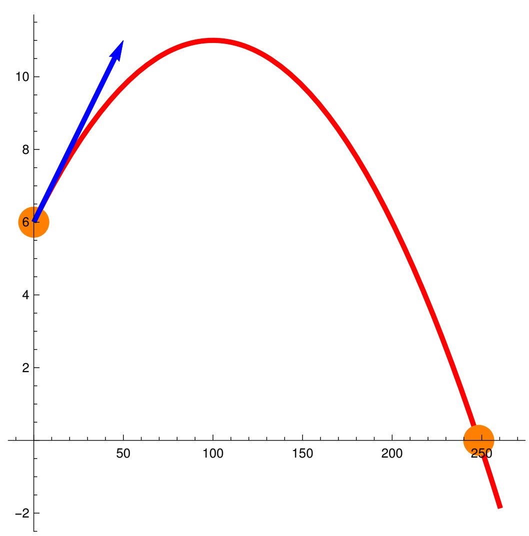
If $\vec{r}\,'(t) \neq \vec{0}$, any vector parallel to $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

The **addition rule** in one dimension $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.
The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ at all times as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

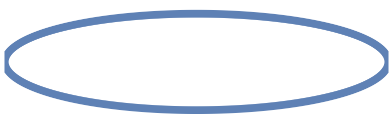
The **free fall** is the case when acceleration is constant. In particular, if $\vec{r}\,''(t) = [0, 0, -10]$, $\vec{r}'(0) = [0, 1000, 2], \vec{r}(0) = [0, 0, h]$, then $\vec{r}(t) = [0, 1000t, h + 2t - 10t^2/2]$.

If $r\,''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2$.



Unit 5: Curves

1 Match the curves with the equations



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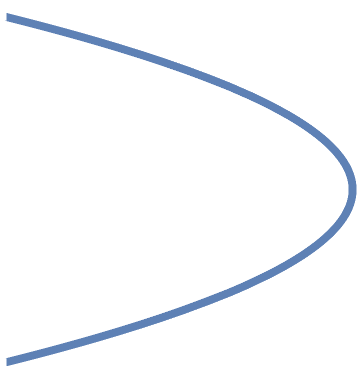

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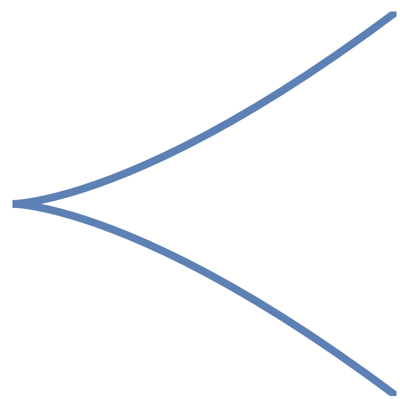

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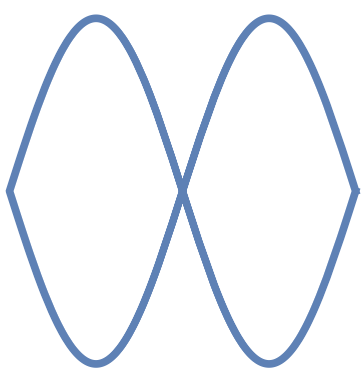

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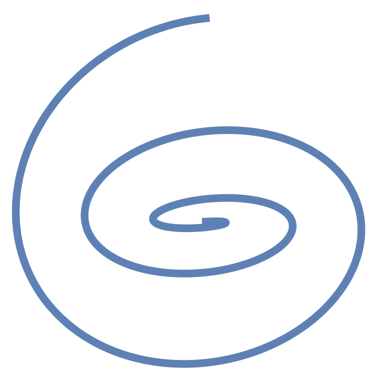


6

Enter number 1-6 of the figure	Parametric equation of the curve
	$\vec{r}(t) = [\cos^2(t), \sin(t)]$
	$\vec{r}(t) = [t^2, t^3]$
	$\vec{r}(t) = [4\cos(t), \cos(\pi/2 - t)]$
	$\vec{r}(t) = [t\sin(t), t^2\cos(t)]$
	$\vec{r}(t) = [\sin(t), t]$
	$\vec{r}(t) = [\lvert t\rvert, \sin(t)]$

6: Arc Length and Curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\;dt$ is called the **arc length of the curve**. Written out in three dimensions, this is $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\;dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = [R\cos(t), R\sin(t)]$ parameterized by $0 \leq t \leq 2\pi$ is 2π because the speed $|\vec{r}'(t)|$ is constant R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = [\cos(t), \sin(t), t]$ has velocity $\vec{r}\,'(t) = [-\sin(t), \cos(t), 1]$ and constant speed $|\vec{r}\,'(t)| = [-\sin(t), \cos(t), 1] = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = [t, \log(t), t^2/2]$$

for $1 \leq t \leq 2$? **Answer:** Because $\vec{r}\,'(t) = [1, 1/t, t]$, we have $\vec{r}\,'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\;dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$.

4 Find the arc length of the curve $\vec{r}(t) = [3t^2, 6t, t^3]$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = [\cos^3(t), \sin^3(t)], 0 \leq t \leq 2\pi$? **Answer:** We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. The absolute value forces us to split the integral into 4 intervals. Since $\int_0^{\pi/2} \sin(2t)\;dt = 1$, we have $\int_0^{2\pi} (3/2)|\sin(2t)|\;dt = (3/2)4 = 6$.

6 Find the arc length of $\vec{r}(t) = [t^2/2, t^3/3]$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. The speed is $|\vec{r}(t)| = \sqrt{t^2 + t^4}$. Because $\int x\sqrt{1 + x^2}\;dx = (1 + x^2)^{3/2}/3$, the arc length integral can be evaluated using substitution by as $\int_{-1}^1 |t|\sqrt{1 + t^2}\;dx = 2\int_0^1 t\sqrt{1 + t^2}\;dt = 2(1 + t^2)^{3/2}/3|_0^1 = 2(2\sqrt{2} - 1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = [t + \sin(t), \cos(t)]$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r}\,'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\;dt$. A **substitution** $t = 2u$ gives $L = \int_0^\pi \sqrt{2 + 2\cos(2u)}\;2du = \int_0^\pi \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\;2du = \int_0^\pi \sqrt{4\cos^2(u)}\;2du = 4\int_0^\pi |\cos(u)|\;du = 8$.

8 Compute the arc length of the **catenary** $\vec{r}(t) = [t, e^t + e^{-t}]$ on an interval $[a,b]$ can be computed as $e^b - e^a - e^{-b} + e^{-a}$. By the way, $(e^t + e^{-t})/2$ is called the hyperbolic cosine and denoted by $\cosh(t)$.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = [\cos(t^2), \sin(t^2)]$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\ dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 We do not always have a closed formula for the arc length of a curve. The length of the **Lissajous figure** $\vec{r}(t) = [\cos(3t), \sin(5t)]$ leads to $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\ dt$ which needs to be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the magnitude of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve turns sharply. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. If $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.
Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = [t, f(t)]$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by $\vec{N}$ and $\vec{B}$ is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because B is automatically normal to T and N, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We prove this in class.

Unit 6: Arc length

In this worksheet, we find the arc length of the **cycloid**

$$\vec{r}(t) = \begin{bmatrix} t - \sin(t) \\ \cos(t) \end{bmatrix}$$

from 0 to 2π. The curve is the solution to the famous **Brachistochrone problem**, the curve along which a ball descents fastest.

1 Compute the velocity $\vec{r}\,'(t)$.

2 Verify that $|\vec{r}\,'(t)| = \sqrt{2 - 2\cos t}$.

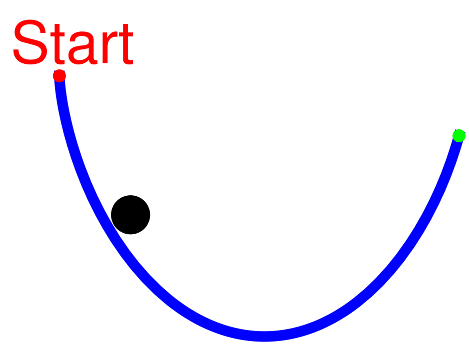
3 Use the double angle formula identity

$$2 - 2\cos(t) = 4\sin^2(\frac{t}{2}) .$$

to find the arc length

$$\int_0^{2\pi} |\vec{r}\,'(t)| \, dt .$$

Johann Bernoulli asked the Brachistochrone problem in 1696. The problem marks the start of a mathematical area called the **calculus of variations** in which one extremizes functions on infinite dimensional spaces.
Cycloids are curves traced by your feet, when you bike. It is a natural curve because it combines linear and circular motion.



About integration techniques

When looking at arc length integrals, basic integration techniques come back. Can you solve the following problems?

1 $\int_0^{2\pi} x\sin(x)\, dx$

2 $\int_0^{\sqrt{\pi}} 2x\sin(x^2)\, dx$.

3 $\int_0^1 \frac{1}{1+x^2}\, dx$.

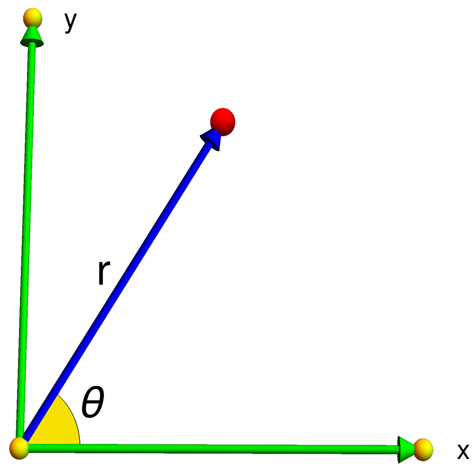
4 $\int_0^{2\pi} \cos^2(x)\, dx$

1

[1] 1) -2π, 2) 2, 3) $\pi/4$. 4) π (use double angle formula)

7: Polar and cylindrical coordinates

We first look at polar coordinates in two dimensions:

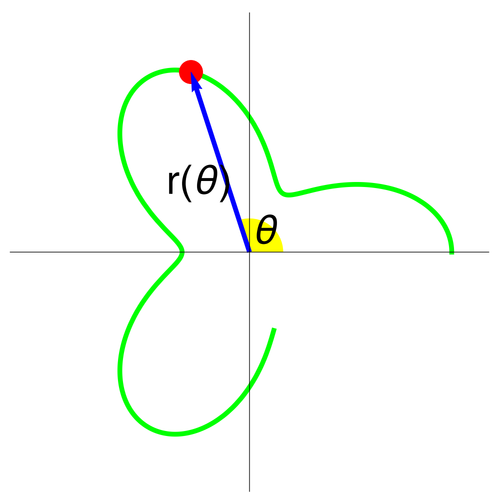


A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$ leading to the relation $(x, y) = (r\cos(\theta), r\sin(\theta))$.

The formula $\theta = \arctan(y/x)$ defines the angle θ only up to an addition of π as (x, y) and $(-x, -y)$ have the same θ value. It is custom to let $\arctan(y/x)$ be in $(-\pi/2, \pi/2]$ for $x > 0$ and define it to be $\pi/2$ for the positive y axes and $\arctan(y/x) + \pi$ for $x < 0$ and equal to $-\pi/2$ on the negative y-axes. For $(x, y) = (0, 0)$, the polar angle θ is not defined.

Some curves can be described nicely in polar coordinates. For example, the unit circle is

$$r = 1 .$$



A curve given in polar coordinates as $r(\theta) = f(\theta)$ is called a **polar curve**. It can in Cartesian coordinates be described as $\vec{r}(t) = [f(t)\cos(t), f(t)\sin(t)]$.

1 The curve

$$\vec{r}(t) = [t\cos(t), t\sin(t)] = [x(t), y(t)]$$

describes a spiral. How can one describe this curve in polar coordinates? We see that $t = \theta$ is the angle and that the distance to the origin is $t = \theta$. Therefore, the curve is

$$r = \theta$$

2 What is the curve given in polar coordinates as

$$r = |2\sin(\theta)| \ .$$

Solution: Let us ignore the absolute value for a moment and multiply both sides with r. This gives

$$r^2 = 2r\sin(\theta)$$

and can be written as $x^2+y^2 = 2y$ which is $x^2+y^2-2y+1 = 1$. A completion of the square shows that this curve is a circle of radius 1 centered at $(0,1)$. Since we have the absolute value, we get an other circle of radius 1 centered at $(0,-1)$. This is when θ is between π and 2π.

Writing a point

$$(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$$

in the form

$$(r,\theta,z)$$

means using **cylindrical coordinates**. It is just using polar coordinates in the xy-plane and keeping the variable z.

Here are some surfaces described in cylindrical coordinates:

3 $r = 1$ is a **cylinder**,

4 $r = |z|$ is a **double cone**

5 $\theta = 0$ is a **half plane**

6 $r = \theta$ is a **rolled sheet of paper**

7 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

Remark which is not relevant for this course but which will come up in 21b:
A point in $\mathbb{R}^2$ can also be represented as a **complex number** $z = x + iy \in \mathbb{C}$. The symbol i means now $\sqrt{-1}$. This is not only notational convenience. Complex numbers can be added and multiplied like other numbers and while $\mathbb{R}^2 = \mathbb{C}$, the later has a **multiplicative structure**. The basic rule to know is $i^2 = -1$ so that $(a+ib)(c+id) = ac - bd + i(ad+bc)$. An important observation of Euler is a link between the exponential and trigonometric functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This implies for $\theta = \pi$ the **nicest formula** in mathematics [1]

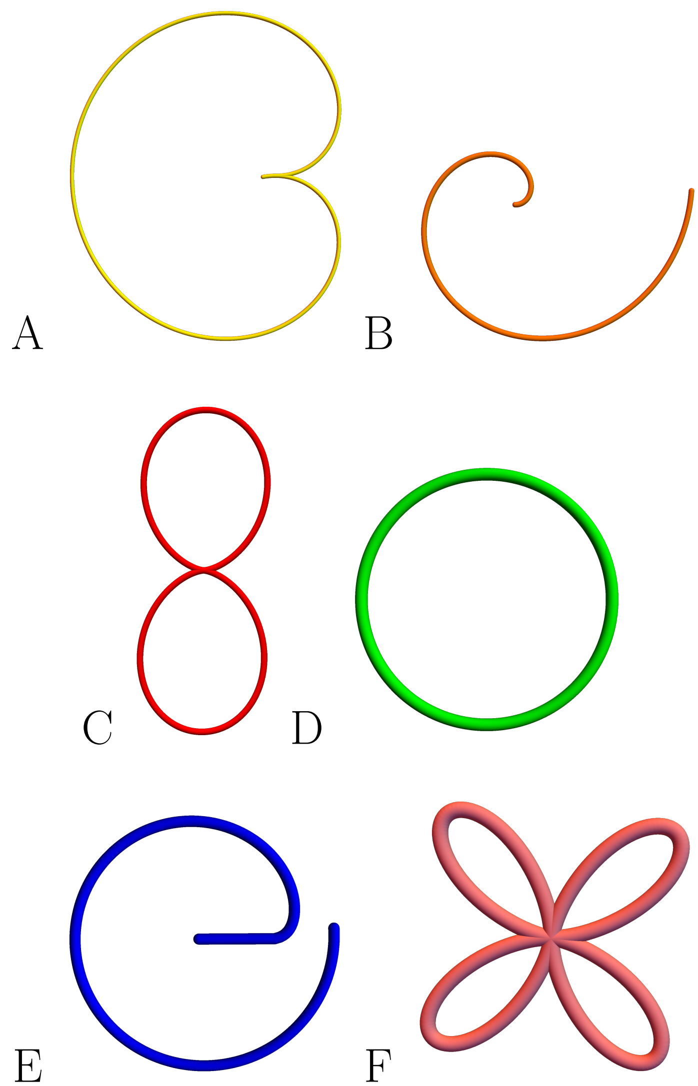
$$\boxed{e^{i\pi} + 1 = 0}$$

It combines "calculus" in the form e, "geometry" in the form of π, "algebra" in the form of i, the additive unit 0 and the multiplicative unit 1.

[1] D. Wells, Which is the most beautiful?, Mathematical Intelligencer, 1988

Unit 7: Match the polar curves

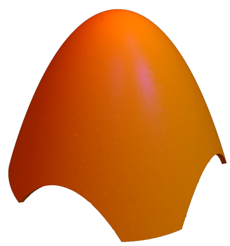
1 Match the polar curves (curves given in polar coordinates) with their formal description



- $r = \theta$
- $r = \theta(2\pi - \theta)$
- $r = \theta^{1/10}$
- $r = \sin^2(\theta)$
- $r = 2$
- $r = \sin^2(2\theta)$

Unit 7: Cylindrical coordinates

1 Match the following surfaces with their descriptions given in cylindrical coordinates



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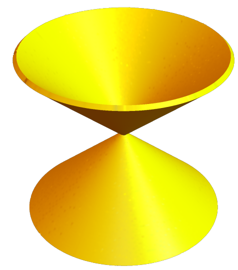

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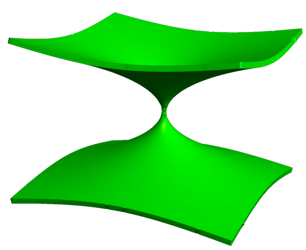

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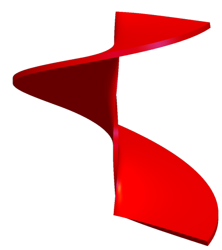

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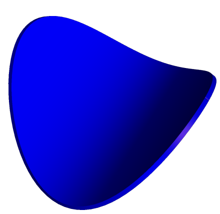
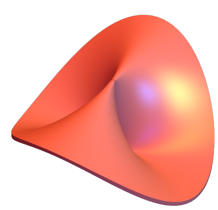


E



F

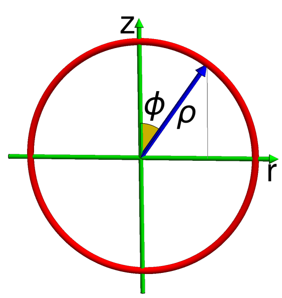
- $r - z^2 = 0$
- $r^2 + z = 1$
- $z = \sin(\theta)^2$
- $z = r^2 \cos(\theta)^2$
- $r^2 - z^2 = 0$
- $z = \theta$

8: Spherical coordinates

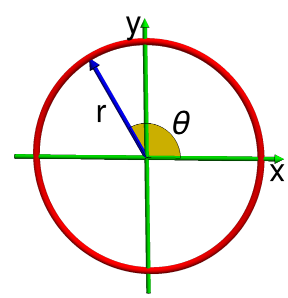
Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ. The first angle θ is the polar angle in polar coordinates of the xy coordinates and ϕ is the angle between the vector $\vec{OP}$ and the z-axis. The relation is

$$(x, y, z) = (\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) .$$

There are two important figures to see the connection. The distance to the z-axis $\boxed{r = \rho \sin(\phi)}$ and the height $z = \rho \cos(\phi)$ can be read off by the left picture the rz-plane, the coordinates $x = r \cos(\theta), y = r \sin(\theta)$ can be seen in the right picture the xy-plane.



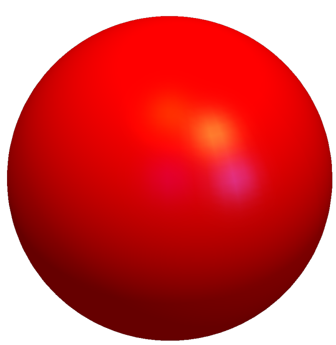
$$\boxed{\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) \\ y &= \rho \sin(\phi) \sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}}$$

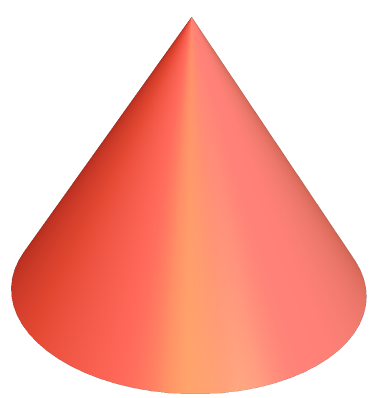


Here are some surfaces described in spherical coordinates.

1 $\rho = 1$ is a **sphere**.

2 The surface $\phi = 3\pi/4$ is a **single cone**.



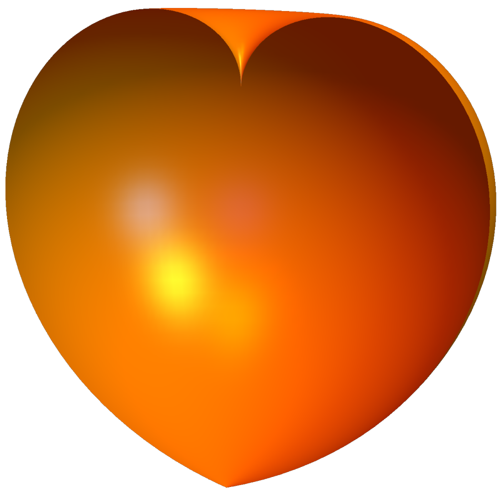


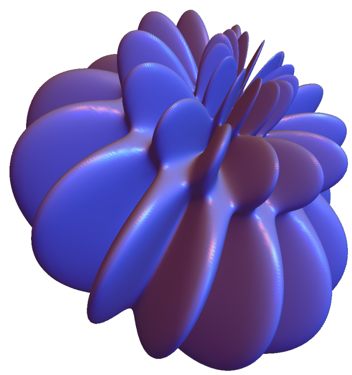
3 The surface $\phi = \pi/2$ is the xy-plane.

4 The surface $\sin(\theta) = \cos(\theta)$ is a **plane** if we take the liberty to allow on the z-axes any θ value.

5 $\rho = \phi$ is an **apple shaped surface**. We plot only half of it

6 $\rho = 2 + \cos(13\theta)\sin(\phi^2) + Cos[s]$ is an example of a **jelly fish** The radius ρ depends on the two angles. Jelly fish are cool as they do not die!





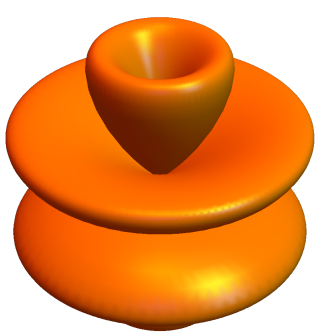
7 Match the two surfaces below with either

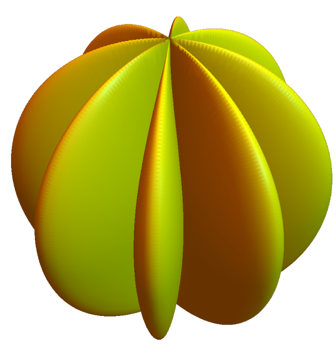
$$\rho = |\sin(3\phi)|$$

and

$$\rho = |\sin(3\theta)|$$

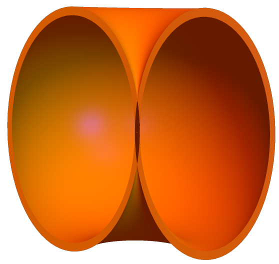
in spherical coordinates (ρ, θ, ϕ).





Unit 8: Match spherical surfaces

1 Match the surfaces given in spherical coordinate descriptions: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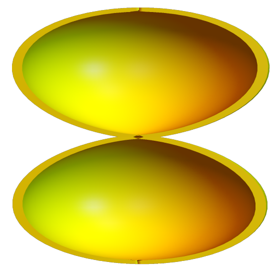
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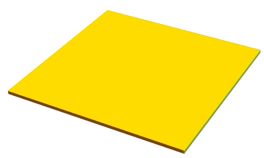
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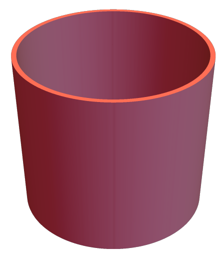
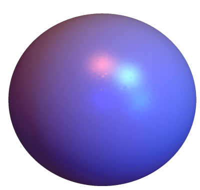
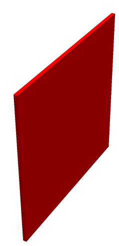

D


E


F

- $\rho = \sin(\phi)$
- $\rho = \cos^2(\phi)$
- $1 = \sin(\phi)$
- $1 = \rho \sin(\phi)$
- $1 = \sin(\rho)$
- $1 = \cos(\theta)$

Unit 8: Spherical coordinates

1 What is the point $(-1, 1, 1)$ in spherical coordinates?

2 What is the point $(0, -3, -3)$ in spherical coordinates?

3 What point does $(\rho, \theta, \phi) = (3, \pi, \pi/4)$ represent?

4 Describe the surface $\rho = \cos(\theta)\sin(\phi)$.

5 Describe the surface $\rho = \cos(\phi)$.

9: Parametrizations of surfaces

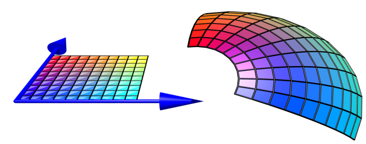
Planes can be described either by implicit equations $x + y + z = 1$ or by parametrization $\vec{r}(t, s) = [1 + t + s, -t, -s]$. More general surfaces like graphs $z = f(x, y)$ can be parametrized as $\vec{r}(x, y) = [x, y, f(x, y)]$ matching a point in the xy-plane with a point in space. Today, we generalize this:

A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix},$$

where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables.

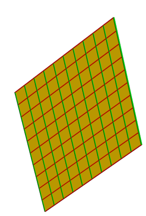
Because two parameters u and v are involved, the map $\vec{r}$ from is also called ***uv*-map**. A **parametrized surface** is the image of the uv-map. The domain of the uv-map is called the **parameter domain**.



If the first parameter u is kept constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, for constant v, the map $u \mapsto \vec{r}(u, v)$ traces a curve on the surface. These curves are called **grid curves**.

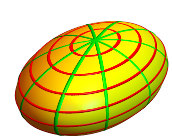
A computer draws surfaces using grid curves. The world of parametric surfaces is intriguing. It can be explored with the help of a computer and can be hard. Keep in mind the following four important examples. They cover a wide range of cases.

1) **Planes**. The parametric description $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$ has already been covered. The implicit equation $ax + by + cz = d$ can be obtained by computing the normal vector $\vec{n} = \vec{v} \times \vec{w} = [a, b, c]$. Conversely, if we are given a plane like $x - y + 3z = 6$, we can find three points like $A = (0, -6, 0), B = (6, 0, 0), C = (0, 0, 2)$, form the vectors $\vec{v} = \vec{AB} = [6, 6, 0], \vec{w} = [0, 6, 2]$ and parametrize



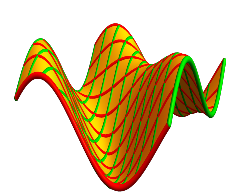
$$\vec{r}(s, t) = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} + s \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix}.$$

2) **Spheres**. Remember spherical coordinates? $\rho = 1$ was a sphere. If we vary the other parameters θ, ϕ we get $\vec{r}(\theta, \phi) = [\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)]$. We can modify this to get parametrizations of ellipsoids. An example is $\vec{r}(\theta, \phi) = [3\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 5\cos(\phi)]$.

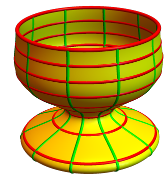


3) **Graphs** can be parametrized as

$$\vec{r}(x, y) = [x, y, f(x, y)].$$



The parametrization of the paraboloid $z = x^2 + y^2$ is $[x, y, x^2 + y^2]$. The graphs can also be turned. The parametrization of the surface $y = \cos(xz)$ for example is $\vec{r}(x, z) = [x, \cos(xz), z]$.



4) **Surfaces of revolution** parametrizations are based on cylindrical coordinates. $\vec{r}(u,v) = [g(v)\cos(u), g(v)\sin(u), v]$
It belongs to the surface $r = g(z)$ which can be written as $x^2 + y^2 = g(z)^2$.

1 **Problem:** Draw the surface $\vec{r}(u,v) = [v^5\cos(u), v^5\sin(u), v]$.
Solution. It is a surface of revolution. We have $r = v^5 = z^5$. We see that $r = z^5$. A good way to start is to draw $z = r^{1/5}$ in the rz-plane, then imagine rotating the graph around the z-axis.

2 **Problem:** Find a parametrization for the plane through the three points $P = (3,7,1)$,$Q = (6,2,1)$ and $R = (0,3,4)$.
Solution. This is a problem we have encountered before Take $\vec{r}(s,t) = \vec{OP} + s\vec{QP} + t\vec{RP}$. $\vec{r}(s,t) = [3 - 3s - 3t, 7 - 5s - 4t, 1 + 3t]$.

3 **Problem:** Parametrize the lower half of the ellipsoid $x^2/4 + y^2/9 + z^2/25 = 1, z < 0$.
Solution. We could solve for z and get $\vec{r}(u,v) = [u, v, -\sqrt{25 - 25u^2/4 - 25v^2/9}]$. This is one solution. It works here because the lower half of the ellipsoid is the graph of a function $f(x,y)$. It would not work for the entire ellipsoid. Better is to deform the sphere parametrization and get

$$\vec{r}(\theta,\phi) = \begin{bmatrix} 2\sin(\phi)\cos(\theta) \\ 3\sin(\phi)\sin(\theta) \\ 5\cos(\phi) \end{bmatrix}$$

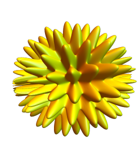
and restrict ϕ to $[\pi/2, \pi]$.

4 **Problem:** Parametrize the upper half of the hyperboloid $(x-1)^2 + y^2/4 - z^2 = -1$.
Solution. First take the round hyperboloid $r^2 = z^2 - 1$ is parametrized by $\vec{r}(\theta,z) = [\sqrt{z^2-1}\cos(\theta), \sqrt{z^2-1}\sin(\theta), z]$. Deform this now to get for $x^2 + y^2/4 - z^2 = -1$ the parametrization $\vec{r}(\theta,z) = [\sqrt{z^2-1}\cos(\theta), 2\sqrt{z^2-1}\sin(\theta), z]$. Now move still the x coordinate to get $\vec{r}(\theta,z) = [1 + \sqrt{z^2-1}\cos(\theta), 2\sqrt{z^2-1}\sin(\theta), z]$.

5 The surface given in spherical coordinates as $\rho = (2 + \sin(13\theta) + \sin(17\phi))$ is the **hedgehog**. Parametrize it in the form $\vec{r}(u,v)$. Solution: Of course, we could use also other variables like θ and ϕ but as we have been forced to use u and v, lets take $u = \theta$ and $v = \phi$. We write

$$\vec{r}(u,v) = (2 + \sin(13u) + \sin(17v)) \begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix} .$$

Note that it was a bit more convenient to take the radius expression ρ outside the bracket so that we do not have to rewrite it. You recognize then also better the spherical coordinates as the expression to the left is ρ.



Unit 9: Parametrized surfaces

In order to study parametrized surfaces, it is helpful to look at **grid curves**. These are the curves obtained if one parameter is kept fixed. For a sphere, the **circles of latitudes** and **half circles of longitudes** are the grid curves. The **equator** is the circle of latitude of 0 which corresponds to $\phi = \pi/2$. The conversion is latitude $= \pi/2 - \phi$. The sphere is parametrized by

$$\vec{r}(u, v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] \ .$$

We analyze now the surface

$$\vec{r}(u, v) = [u\cos(v), u\sin(v), v] \ .$$

1 What are the grid curves where u is constant, like $u = 1$?

2 What are the grid curves where v is constant, like $v = 0$?

3 Can you draw the surface?

4 Let us change the parametrization to

$$\vec{r}(u, v) = [u\cos(v), u\sin(v), u] \ .$$

What surface is this now?

10: Functions

A function $f(x,y)$ with domain R is called **continuous at a point** $(a,b) \in R$ if $f(x,y) \to f(a,b)$ whenever $(x,y) \to (a,b)$. The function f is **continuous on** R, if f is continuous for every point (a,b) on R. Sometimes, we can extend the domain R to make it continuous there too. The function $f(x,y) = \sin(x)y/x$ for example can be assigned the value y on the axes $x = 0$ as l'Hopital shows.

1 The function $f(x,y) = x^2 + y^4 + xy + \sin(y + \sin\sin\sin\sin(x)^2)$ is continuous on the entire plane. It is built up from functions which are continuous using addition, multiplication or composition of functions which are all continuous.

2 $f(x,y) = 1/(x^2 + y^2)$ is continuous everywhere except at the origin, where it is not defined. We can not extend the value as the value at $(0,0)$ would have to be arbitrarily large.

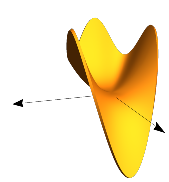
3 $f(x,y) = y + \sin(x)/|x|$ is continuous except at $x = 0$. At every point $(0,y)$ it is discontinuous. $f(1/n,y) \to y+1$ and $f(-1/n,y) \to y-1$ for $n \to \infty$.

4 $f(x,y) = \sin(1/(x+y))$ is continuous except on the line $x + y = 0$.

5

$$f(x,y) = (x^4 - y^5)/(x^2 + y^2)$$

is continuous at $(0,0)$. Use polar coordinates to see that it is $r^2\cos^2(t)$.



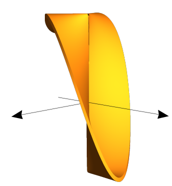
6 The function $f(x) = e^{-1/x^2}$ is continuous everywhere. Actually, even all derivatives are continuous and are zero at 0. Still, the function is not constant zero.

7 There are three sources for discontinuous behavior: there can be **jumps**, there can be **poles**, or the function can **oscillate**. An example of a jump appears with $f(x) = \sin(x)/|x|$, a pole example is $g(x) = 1/x$ leads to a vertical asymptote and the function going to infinity. An example of a function discontinuous due to oscillations is $h(x) = \sin(1/x)$. Its graph is the **devil's comb**.

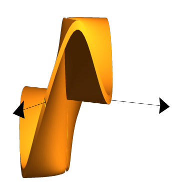
There are two handy tools to discover a discontinuities:
1) Use polar coordinates with coordinate center at the point to analyze the function.
2) Restrict the function to one dimensional curves and check continuity on that curve, where one has a function of one variables.

8 Determine whether the function $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is continuous at $(0,0)$. **Solution** Use polar coordinates to write this as $\sin(r^2)/r^2$ which is continuous at 0 (apply l'Hopital twice if you want to verify this).

9 Is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous at $(0,0)$? **Solution** Use polar coordinates to see that this $\cos(2\theta)$. We see that the value depends on the angle only. Arbitrarily close to $(0,0)$, the function takes any value from -1 to 1.



10 Is the function $f(x,y) = \frac{x^2 y}{x^4+y^2}$ continuous?
Solution. Look on the parabola $x^2 = y$ to get the function $x^4/(2x^4) = 1/(2x^2)$ which is not continuous at 0. This example is **shocking** because it is continuous through each line through the origin: if $y = ax$, then $f(x,ax) = ax^3/(x^4 + a^2x^2) = ax/(x^2 + a^2)$. This converges to 0 for $x \to 0$ as long as $a \neq 0$. If $a = 0$ however, we have $y = 0$ and $f = 0/x^4$ which can be continuously extended to $x = 0$ too.



11 What about the function

$$f(x,y) = \frac{xy^2 + y^3}{x^2 + y^2}$$

Solution. Use polar coordinates and write $r^3 \sin^2(\theta)(\cos(\theta) + \sin(\theta)/r^2 = r \sin^2(\theta)(\cos(\theta) + \sin(\theta)$ which shows that the function converges to 0 as $r \to 0$.

12 Is the function $f(x,y) = \frac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$ continuous everywhere? **Solution.** Use polar coordinates to see that this is $\sin(r)/r$. This function is continuous at 0 by Hôpital's theorem.

Unit 10: Functions

1 Compute the limit $\tan(t)/t$ for $t \to 0$ using l'Hopital.

2 Compute the limit $(x^2-1)/(x^2+1)$ for $x \to \infty$ using l'Hopital.

3 Compute the limit $\sin^2(x)/x^2$ for $x \to 0$ using l'Hopital

4 Is the function $f(x,y) = (x^4+y^4)/(x^2+y^2)^2$ with $f(0,0) = 0$ continuous?

5 Is the curve

$$\vec{r}(t) = [\cos(t), \sin(t), \sin(t)/t]$$

with $\vec{r}(0) = [1,0,1]$ continuous?

6 Can you plot the surface

$$\vec{r}(\theta, z) = [\cos^3(\theta), \sin^3(\theta), z] \ ?$$

11: Partial derivatives

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

We also write $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. [1]

1 For $f(x,y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation** for the unknown function $f(x,y)$ involving partial derivatives. The vector $[f_x, f_y]$ is called the gradient.

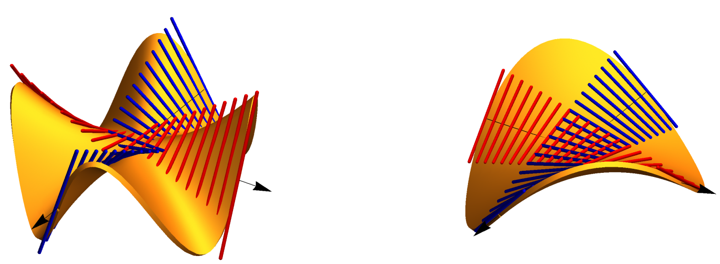
Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$hf_x(x,y) = f(x+h,y) - f(x,y)$
$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x+h,y+h) - (f(x+h,y) - f(x,y))$

$hf_y(x,y) = f(x,y+h) - f(x,y)$.
$h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$

No limits were taken. We established an identity which holds for all $h > 0$, the discrete derivatives f_x, f_y satisfy $f_{xy} = f_{yx}$. It is a "quantum Clairaut" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, the limit $h \to 0$ leads to the classical Clairaut's theorem. The quantum Clairaut theorem holds for **any** functions $f(x,y)$ of two variables. Not even continuity is needed.



2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6y^{10}\cos(y)$. Hint: No need not compute, just think.

3 The continuity assumption for f_{xy} is necessary. The example $f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$ contradicts Clairaut's theorem:

[1] $\partial_x f, \partial_y f$ were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences".

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$, $f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

$f_x(x_0, y_0)$ measures the slope when slicing the graph $z = f(x,y)$ in the x-direction.
f_{xx} measures the concavity when slicing the graph in the x-direction.
f_{xy} measures how the x slope changes when you move in the y direction.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

Here are two examples of partial differential equations. We will look at them in more detail next time and try to make sense what they mean.

4 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $f(t,x) = \sin(x-t) + \sin(x+t)$ satisfies the wave equation.

5 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

6 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $f(t,x) = \sin(x-t) + \sin(x+t)$ satisfies the wave equation.

7 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

8 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x,y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation. Such functions are called **harmonic**.

9 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t,x) = e^{-(x+t)^2}$ satisfy the advection equation.

10 The **Burgers equation** $f_t + ff_x = f_{xx}$ describes waves at the beach which break. The function $f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1 + \sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.

Unit 11: Partial derivatives

Let $f(x, y) = e^{-x^2-y^2}$.

1 Compute $f_x(x, y)$ and especially $f_x(1, 1)$.

2 Compute $f_{xy}(x, y)$ and especially $f_{xy}(1, 1)$

3 Compute $f_{yx}(x, y)$ and especially $f_{yx}(1, 1)$.

And now, if you are brave.

4 Compute $f_{xyx}(x, y)$ and especially $f_{xyx}(1, 1)$.

5 You are told that the result in the previous problem is $-4/e^2$. Can you find the value of $f_{yxx}(1, 1)$?

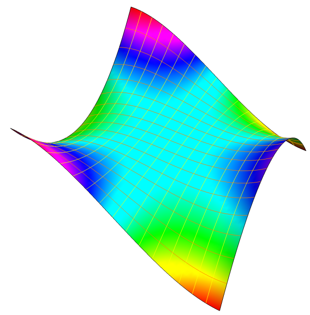
6 What is $f_{xyxxyyy}(1, 2)$ for $f(x, y) = x^2 \sin(\sin(y)) + xy^3 e^{e^{e^x}}$?

Hint for the last one: the answer can be seen without much computation.

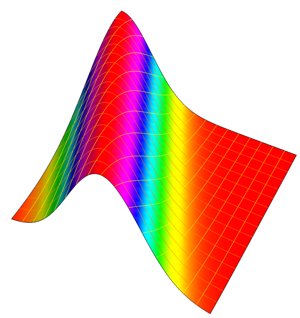
12: Partial Differential Equations

An equation which involves partial derivatives for an unknown function $f(x, y)$ is called a **partial differential equation** or shortly a **PDE**. The topic of PDE's would fill a course by itself. Finding and understanding solutions of PDE's can be difficult. The topic is introduced here in the context of **partial differentiation**. You should be able to verify that a given function $f(t, x)$ satisfies a specific PDE and know some examples. It is useful also to understand what the equations say. In most cases, the equations relate a rate of change in time with a rate of change in space.

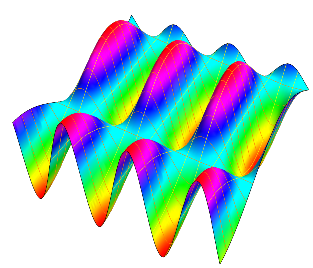
LAPLACE EQUATION. $\boxed{f_{xx} + f_{yy} = 0}$. A stationary temperature distribution on a plate satisfies this equation.
1) Verify that $\boxed{f(x, y) = x^3 - 3xy^2}$ satisfies the Laplace equation.



ADVECTION EQUATION. $\boxed{f_t = f_x}$. Models transport in a one-dimensional medium. It is also called a **transport equation**.
2) Verify that $\boxed{f(t, x) = e^{-(x+t)^2}}$ satisfy the advection equation $f_t(t, x) = f_x(t, x)$.

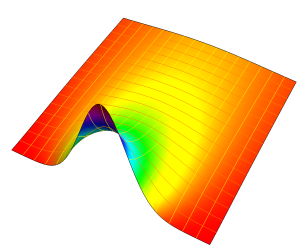


WAVE EQUATION. $\boxed{f_{tt} = f_{xx}}$. For fixed time t, the function $x \mapsto f(t, x)$ describes a string at that time.
3) Verify that $\boxed{f(t, x) = \sin(x - t) + \sin(x + t)}$ satisfies the wave equation $f_{tt}(t, x) = f_{xx}(t, x)$.

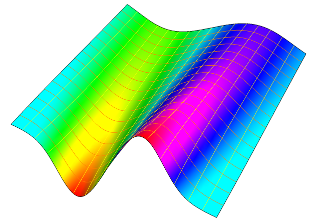


HEAT EQUATION. $\boxed{f_t = f_{xx}}$ For fixed time t, the function $x \mapsto f(t, x)$ is the temperature at the point x. The heat equation is also called **diffusion equation**.

The function $\boxed{f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}}$ satisfies the heat equation.

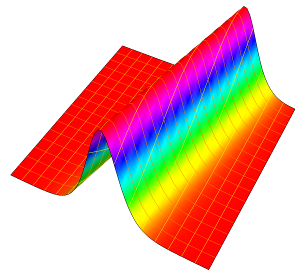


BURGER EQUATION. $\boxed{f_t + f f_x = 0}$ describes waves reaching a beach. In higher dimensions, it leads to the Navier-Stokes equation. One of the millennium (10^6\$) problems is to solve the existence problem in 3D. There are **N-wave** solutions $\boxed{f(t,x) = \frac{x}{t}\frac{\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}}$ Without f_{xx} term, solutions will break (form **shocks**).



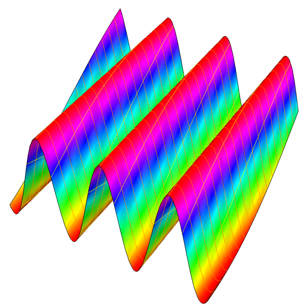
KDV-EQUATION. $\boxed{f_t + 6ff_x + f_{xxx} = 0}$ Describes **water waves** in a narrow channel. First discovered by J. Scott Russel in 1838.

The solution $\boxed{f(t,x) = \frac{a^2}{2}\cosh^{-2}(\frac{a}{2}(x - a^2 t))}$ describes a wave with speed a^2 and amplitude $a^2/2$. It is called a **soliton**. Unlike linear waves, these **nonlinear waves** can travel with different speed: higher waves move faster. Solitons form a fancy research topic.



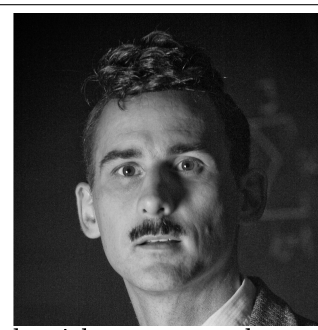
SCHRÖDINGER EQUATION. $\boxed{f_t = \frac{i\hbar}{2m} f_{xx}}$ Describes a free **quantum particle** of mass m.

A solution is $\boxed{f(t,x) = e^{i(kx - \frac{\hbar}{2m}k^2 t)}}$ models a particle with momentum $\hbar k$. The constant i satisfies $i^2 = -1$ (it is an imaginary number), $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.



Note that in this example, the function $f(x,y)$ takes values in the complex plane.

"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.

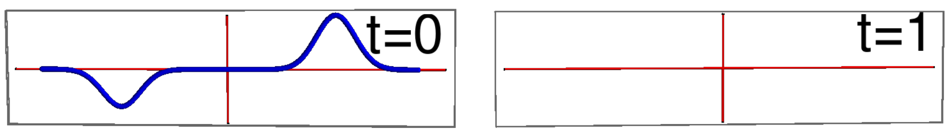


Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed.

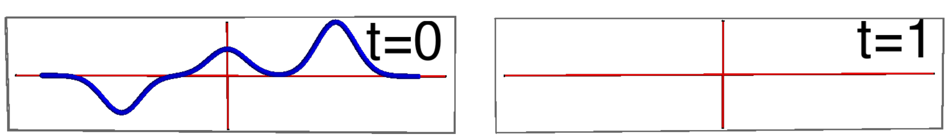
Unit 12: Partial differential equations

1 Verify that $f(t,x) = 2tx + t^2 + x^2$ satisfies the wave equation $\boxed{f_{tt} = f_{xx}}$.

2 Let us assume that $f(t,x)$ satisfies the transport equation $\boxed{f_t = f_x}$ and that $f(0,x)$ is given in the picture to the left. Can you predict, how $f(1,x)$ looks like?



3 Let us assume that $f(t,x)$ satisfies the heat equation $f_t = f_{xx}$ and that $f(0,x)$ is given in the picture to the left. Can you sketch, how $f(1,x)$ looks like?



4 In the Book "In Pursuit of the Unknown", the English mathematician **Ian Stewart** covers 17 equations. Some of the are partial differential equations. One of the following equations appears on the cover of that book. Which one?

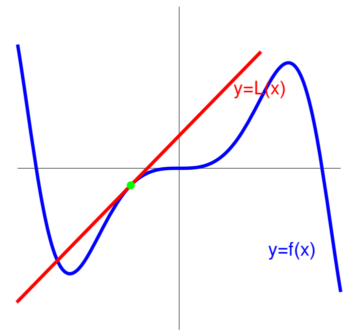
	Transport
	Burgers
	Heat
	Wave

5 Finally see whether you can recognize some other equations or formulas on that book cover illustration.

13: Linearization

The **linear approximation** of a function $f(x)$ at a point a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \; .$$



The graph of the function L is a line close to the graph of f near a. We generalize this to higher dimensions:

The **linear approximation** of $f(x, y)$ at (a, b) is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \; .$$

In three dimensions, the **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) \; .$$

Using the **gradient** $\nabla f(x, y) = [f_x, f_y]$ rsp. $\nabla f(x, y, z) = [f_x, f_y, f_z]$, the linearization can be written as $L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$. By keeping the second variable $y = b$ fixed, we get to a one-dimensional situation, where the only variable is x. Now $f(a, b) + f_x(a, b)(x - a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed, then y is the single variable, then $f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$ is an approximation. Knowing the linear approximations in both the x and y variables, produces the linearization $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

1 What is the linear approximation of the function $f(x, y) = \sin(\pi x y^2)$ at the point $(1, 1)$? We have $(f_x(x, y), y_f(x, y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1, 1)$ equal to $\nabla f(1, 1) = [\pi \cos(\pi), 2\pi \cos(\pi)] = [-\pi, 2\pi]$.

2 Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 \; .$$

3 Find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$. Since $f(1, 1, 1) = 3$, and $\nabla f(x, y, z) = [y+z, x+z, y+x]$, $\nabla f(1, 1, 1) = [2, 2, 2]$. we have $L(x, y, z) = f(1, 1, 1) + [2, 2, 2] \cdot [x-1, y-1, z-1] = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

4 Estimate $f(0.01, 24.8, 1.02)$ for $f(x, y, z) = e^x \sqrt{y} z$.
Solution: take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. **Solution.**The gradient is $\nabla f(x, y, z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) +$

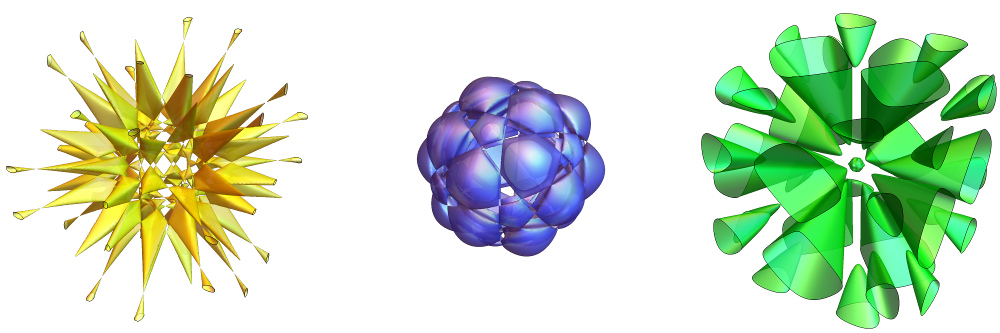
$\nabla f(x_0, y_0, z_0)(x-x_0, y-y_0, z-z_0) = 5+(5, 1/10, 5)(x-0, y-25, z-1) = 5x+y/10+5z-2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5+[5, 1/10, 5]\cdot[0.01, -0.2, 0.02] = 5+0.05-0.02+0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

5 Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2, 4)$. **Solution:** the tangent line is the level curve of the linearization of $L(x, y)$ of $f(x, y) = y - x^2 = 0$ which passes through the point. We compute the gradient $[a, b] = \nabla f(2, 4) = [-g'(2), 1] = [-4, 1]$ and forming $ax + by = -4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$.

6 The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x, y, z) &= (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 \\ &+ 8(x^2 - t^4y^2)(-(t^4x^2) + z^2)(y^2 - t^4z^2)(x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4)\,, \end{aligned}$$

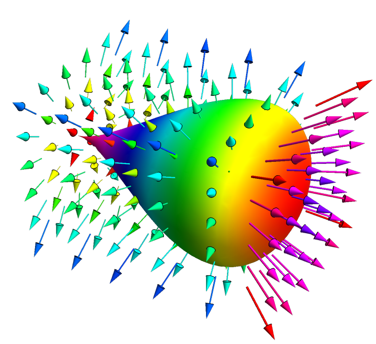
where $t = (\sqrt{5} + 1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5} - 1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x, y, z) = 8$. Find the tangent plane of the later surface at the point $(1, 1, 0)$. **Solution:** We find the level curve of the linearization by computing the gradient $\nabla f(1, 1, 0) = [64, 64, 0]$. The surface is $x + y = d$ for some constant d. By plugging in the point $(1, 1, 0)$ we see that $x + y = 2$.



The quartic surface

$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

7 is called the **piriform**. What is the equation for the tangent plane at the point $P = (2, 2, 2)$ of this pair shaped surface? **Solution.** We get $[a, b, c] = [20, 4, 4]$ and so the equation of the plane $20x + 4y + 4z = 56$, where we have obtained the constant to the right by plugging in the point $(x, y, z) = (2, 2, 2)$.



Lecture 13: Linearization

1 Find the linearization $L(x)$ of the function $f(x) = x^{1/6}$ at $x = 1'000'000$.

2 Estimate $1000100^{1/6}$!

3 Find the linearization of $f(x, y) = \sin(xy^2)$ at $(\pi, 1)$.

4 Estimate $\sin((\pi + 1/10), 1.01^2)$.

14: Chain rule

If f and g are functions of t, then the **single variable chain rule** tells

$$\frac{d}{dt} f(g(t)) = f'(g(t))g'(t) \; .$$

For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$. This **chain rule** can be proven by linearising the functions f and g and verifying the chain rule in the linear case. The rule is useful for finding derivatives like $\arccos'(x)$: write $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))} \arccos'(x) = \sqrt{1-x^2} \arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

1 Find the derivative d/dx $\arctan(x)$. **Solution.** We have $\sin' = \cos$ and $\cos' = -\sin$ and from $\cos^2(x) + \sin^2(x) = 1$. follows $1 + \tan^2(x) = 1/\cos^2(x)$. Therefore $d/dx \tan(\arctan(x)) = 1/\cos^2(\arctan(x)) \tan'(x) = x$ Now $1/\cos^2(x) = 1/(1 + \tan^2(x)$ so that $\tan'(x) = 1/(1 + x^2)$.

Define the **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]$.

If $\vec{r}(t)$ is curve and f is a function of several variables we can build a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, If $\vec{r}(t)$ is a parametrization of a curve in the plane and f is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

The **multi-variable chain rule** is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \; .$$

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \; .$$

Now, the identity

$$\frac{f(x(t+h),y(t+h))-f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h))-f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h))-f(x(t),y(t))}{h}$$

holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t), y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate f with respect to t and y is treated as a constant:

$$\frac{f(\text{ x(t+h) }) - f(x(t))}{h} = \frac{[f(\text{ x(t) + (x(t+h)-x(t)) }) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} \; .$$

Write $H(t) =$ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t) + H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h} \; .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.

2 We move on a circle $\vec{r}(t) = [\cos(t), \sin(t)]$ on a table with temperature distribution $f(x, y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x, y) = [2x, -3y^2]$, $\vec{r}'(t) = [-\sin(t), \cos(t)]$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = [2\cos(t), -3\sin(t)^2] \cdot [-\sin(t), \cos(t)] = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

From $f(x, y) = 0$, we can express y as a function of x. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we get

Implicit differentation: $y' = -f_x/f_y$.

Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x, y, z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x, y)$. We can compute the derivative z_x without actually knowing the function $z(x, y)$. To do so, we consider y a fixed parameter and compute using the chain rule $f_x(x, y, z(x, y))1 + f_z(x, y)z_x(x, y) = 0$. This leads to the following

Implicit differentiation:
$z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ $\qquad$ $z_y(x, y) = -f_y(x, y, z)/f_z(x, y, z)$

3 The surface $f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x, y)$ at the point $(x, y, z) = (2, 1, 1)$. **Solution:** $z_x(x, y) = -f_x(2, 1, 1)/f_z(2, 1, 1) = -4/(2/9) = -18$.

4 How does the chain rule relate to other differentiation rules? **Answer.** The chain rule is universal: it implies single variable differentiation rules like the addition, product and quotient rule in one dimensions:
$f(x, y) = x + y, x = u(t), y = v(t), d/dt(x + y) = f_x u' + f_y v' = u' + v'$.
$f(x, y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x, y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

5 Can one prove the chain rule from linearization and just verifying it for linear functions? **Solution**. Yes, as in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x, y) = ax + by - c$ and if the curve $\vec{r}(t) = [x_0 + tu, y_0 + tv]$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a, b)$ with $\vec{r}\,'(t) = (u, v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

6 Mechanical systems are determined by the energy function $H(x, y)$, which is a function of two variables. The first variable x is the position and the second variable y is the momentum. The equations of motion for the curve $\vec{r}(t) = [x(t), y(t)]$ are called **Hamilton equations**:

$$\begin{aligned} x'(t) &= H_y(x, y) \\ y'(t) &= -H_x(x, y) \end{aligned}$$

In a homework you verify that the energy of a Hamiltonian system is preserved: for every path $\vec{r}(t) = [x(t), y(t)]$ solving the system, we have $H(x(t), y(t)) = const$.

Lecture 14: Chain rule

We walk on in a mountain region of height

$$f(x, y) = x + (2x^2 + 3y^2 - xy)$$

along the curve $\vec{r}(t) = [(1 + t), \sin(t)]$.

1 Find the rate of change of the height $\frac{d}{dt} f(\vec{r}(t))$ at the point $t = 0$ by differentiating the function $t \mapsto f(\vec{r}(t))$ of one variable.

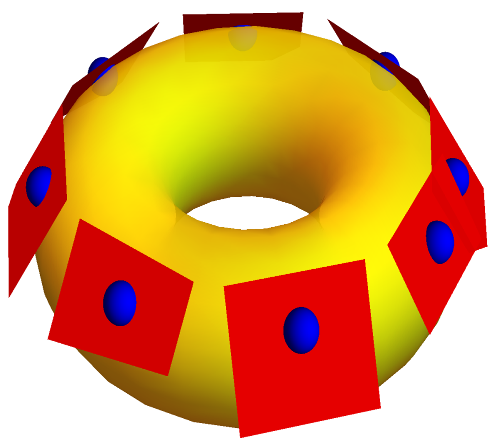
2 Now find the gradient $\nabla f(\vec{r}(0))$ and the velocity vector $\vec{r}\,'(0)$ and use the chain rule to get the derivative again in a different way.

15: Gradient and Tangent

The **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]$ in three dimensions is an important object in multi variable calculus. It is the analog of the derivative $f'(x)$ in one dimensions. The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. The following theorem is important because it provides a crucial link between calculus and geometry. It holds both in two and three dimensions:

Gradient theorem: Gradients are orthogonal to level curves or level surfaces respectively.

Proof: Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$. Because this is true for every curve, the gradient is perpendicular to the surface.

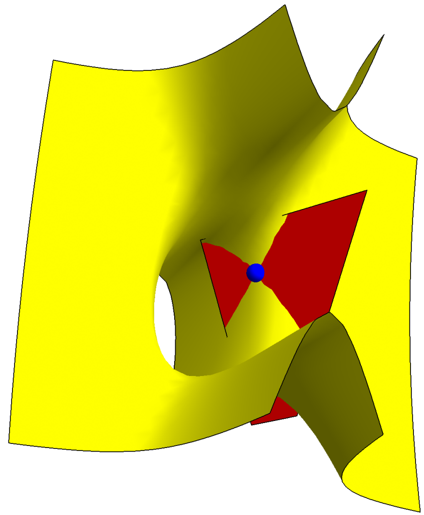


The gradient theorem is useful for example because it allows to get tangent planes and tangent lines very fast, faster than by making a linear approximation:

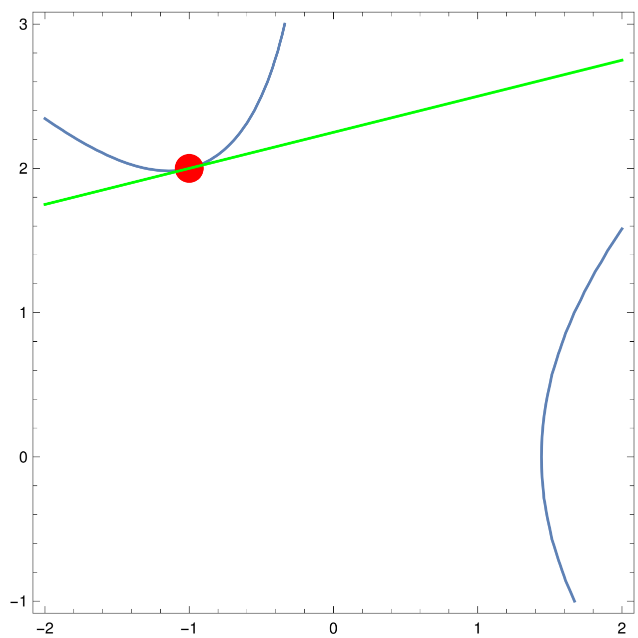
The tangent plane through $P = (x_0, y_0, z_0)$ to a level surface of $f(x,y,z)$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = [a,b,c]$ and d is obtained by plugging in the point P.

The statement in two dimensions is analog.

1 Find the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = [6xy, 3x^2, 2z]$. And $\nabla f(1,1,1) = [6,3,2]$. The plane is $6x+3y+2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x+3y+2z = 11$.



2 **Problem:** Find the tangent line to the curve $f(x,y) = x^3 - y^2x = 3$ at the point $(-1,2)$. **Solution:** The gradient is $\nabla f(x,y) = [3x^2 - y^2, -2yx]$ and $\nabla f(-1,2) = [-1,4]$. The tangent line is $-x + 4y = d$. We get the constant by plugging in the point $(-1,2)$. It is 9. The line is $-x + 4y = 9$.



Lecture 15: Tangent spaces

1 Lets compute the tangent line at $(\pi, 0)$ to the curve $y = \sin(x)$ directly by determining the slope and making sure the line goes through the point.

2 Look at $f(x, y) = y - \sin(x) = 0$. Find the gradient $\nabla f(\pi, 0) = [a, b]$ of f at $(\pi, 0)$. Now find the tangent line again.

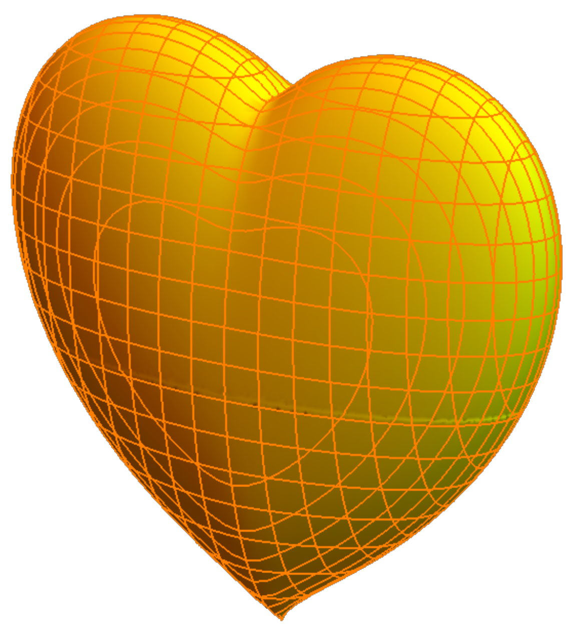
3 Find the tangent plane to the surface $x^2 - y^2 + z^2 = -1$ at the point $(2, 3, 2)$.

4 Find a line perpendicular to the surface $x^2 - y^2 + z^2 = -1$ at the point $(2, 3, 2)$

5 The heart problem (done in class):
Find the tangent plane to the heart

$$(20x^2 + y^2 + z^2 - 1)^3 - (x^2 + y^2)z^3 = 0$$

at the point $(0, 1, 1)$.



16: Directional Derivative

If f is a function of several variables and $\vec{v}$ is a unit vector, then

$$D_{\vec{v}} f = \nabla f \cdot \vec{v}$$

is called the **directional derivative** of f in the direction $\vec{v}$.

The name directional derivative is related to the fact that unit vectors are directions. Because of the chain rule $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$, the directional derivative tells us how the function changes when we move in a given direction. Assume for example that $f(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} f$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $f(x, y)$, then $D_{\vec{v}} f(x, y)$ gives us the slope in the direction $\vec{v}$.

1 If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $\frac{d}{dt} f(\vec{r}(t))$.

2 For $\vec{v} = [1, 0, 0]$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$. The directional derivative generalizes the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f|}$.

Proof. $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}|$.
This implies

The gradient points in the direction where f increases most.

At a point where the gradient ∇f is not the zero vector, the direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

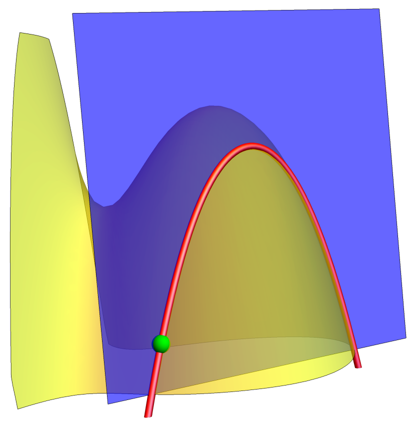
If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

3 You are in an airship at $(1, 2)$ and want to avoid a thunderstorm, a region of low pressure, where pressure is $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure decreases fastest? **Solution:** the pressure gradient is $\nabla p(x, y) = [2x, 4y]$. At the point $(1, 2)$ this is $[2, 8]$. Normalize to get the direction $\vec{v} = [1, 4]/\sqrt{17}$. If you want to head into the direction where pressure is lower, go towards $-\vec{v}$.

Directional derivatives satisfy the same properties then any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f + g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + f D_v(g)$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = [0, 0]$ are called **critical points** and help to understand the function f.

4 Problem. Assume we know $D_v f(1, 1) = 3/\sqrt{5}$ and $D_w f(1, 1) = 5/\sqrt{5}$, where $v = [1, 2]/\sqrt{5}$ and $w = [2, 1]/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f.
Solution: Let $\nabla f(1, 1) = [a, b]$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.



If you should be interested in higher derivatives. We have seen that we can compute f_{xx}. This can be seen as the second directional derivative in the direction $(1, 0)$.

5 The Matterhorn is a famous mountain in the Swiss alps. Its height is 4'478 meters (14'869 feet). Assume in suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the south-east direction $\vec{v} = (1, -1)/\sqrt{2}$. Calculate the rate of change in that direction.
We have $\nabla f(x, y) = [-2x, -2y]$, so that $(20, -20) \cdot (1, -1)/\sqrt{2} = 40/\sqrt{2}$. This is a place, where you climb $40/\sqrt{2}$ meters up when advancing 1 meter forward.

We can also look at higher derivatives in a direction. It can be used to measure the concavity of the function in the $\vec{v}$ direction.

The second directional derivative in the direction $\vec{v}$ is $D_{\vec{v}} D_{\vec{v}} f(x, y)$.

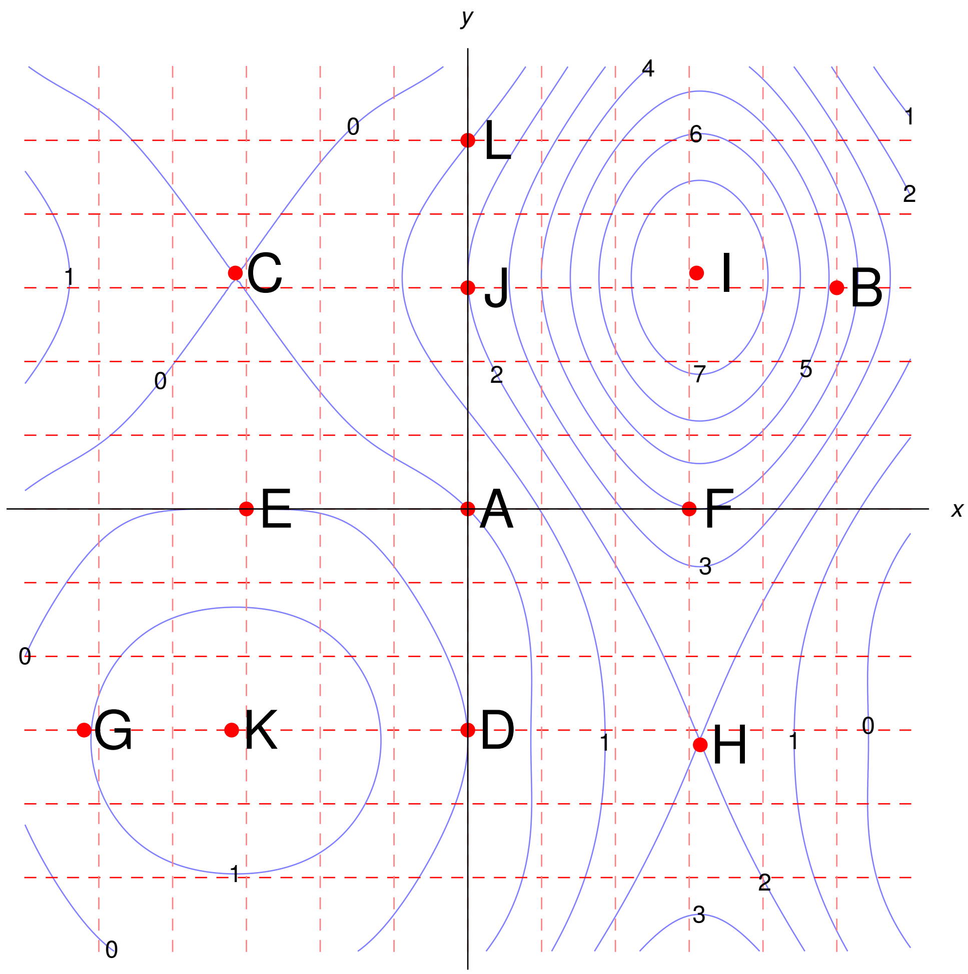
6 For the function $f(x, y) = x^2 + y^2$ the first directional derivative at a point in the direction $[1, 2]/\sqrt{5}$ is $[2x, 2y] \cdot [1, 2] = (2x + 4y)/\sqrt{5}$. The second directional derivative in the same direction is $[2, 4] \cdot [1, 2]/5 = 6/5$. This reflects the fact that the graph of f is concave up in the direction $[1, 2]/5$.

Lecture 16: Directional derivative

a) You see a contour map of a function $f(x, y)$. Draw the gradient at each of the 5 points A-E. If the gradient should be zero, just mark the point with a bubble.

b) Check the boxes which apply. It is in principle possible that more than one box has to be checked in a row or column or that no box needs to be checked in a row or column.

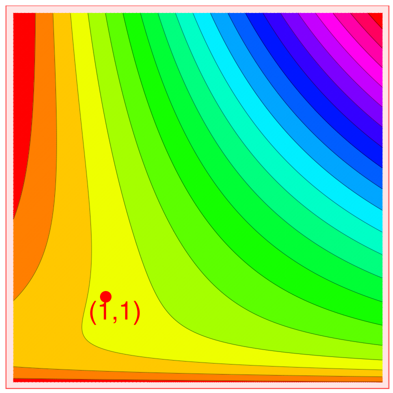
	A	B	C	D	E	F	G	H	I	J	K	L
Maximal steepness among A-L												
$f_x = 0, f_y \neq 0$												
$f_y = 0, f_x \neq 0$												
$D_{[1,-1]/2^{1/2}} f = 0$												
$D_{[1,1]/2^{1/2}} f = 0$												
$f_{xx} > 0, f_{yy} < 0, f_x = 0, f_y = 0$												
$f_{xx} < 0, f_{yy} > 0, f_x = 0, f_y = 0$												



The van der Waals equation for real gases is

$$\boxed{(p + \frac{a}{V^2})(V - b) = RT}$$

where $R = 8.314 J/K$mol is a constant called the **Avogadro number**. This law was relates the **pressure** p, the **volume** V and the **temperature** T of a gas. The constant a is related to the molecular interactions, the constant b to the finite rest-volume of the molecules. For $a = b = 0$, it becomes the ideal gas law $PV = RT$.



The **ideal gas** law $pV = RT$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(V, p) = c$ tell much about the properties of the gas. The **reduced van der Waals law**

$$\boxed{T(V, p) = (p + \frac{3}{V^2})(3V - 1)}$$

is obtained by scaling p, T, V depending on the gas. It has the advantage that it does no more contain constants.

1 What is the gradient of $T(V, p)$ at $(1, 1)$? Check in the figure below whether the gradient is orthogonal to the level curve through $(1, 1)$.

2 Calculate the directional derivative of $T(V, p)$ at the point $(V, p) = (1, 1)$ into the direction $(3, 4)/5$.

17: Extrema

Our goal is to **maximize** or **minimize** a function $f(x,y)$ of two variables. As in one dimension, in order to look for maxima or minima, we consider points, where the "derivative" is zero. In higher dimension, the gradient plays the role of the derivative.

A point (a,b) is called a **critical point** of $f(x,y)$ if $\nabla f(a,b) = [0,0]$.

Critical points are candidates for extrema because at critical points, all directional derivatives $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ are zero. We can not increase the value of f by moving into any direction.

[1]

1 To find the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$, compute the gradient $\nabla f(x,y) = [4(x^3 - y), 4(y^3 - x)]$. Solving the equations gives $(0,0), (1,1), (-1,-1)$.

2 For $f(x,y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x,y) = [2x\cos(x^2+y), \cos(x^2+y)+1]$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $...(0,-\pi),(0,\pi),(0,3\pi),....$

3 For $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$, which has a graph looking like a volcano, the gradient $\nabla f = [2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2)]e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2+y^2=1$. This function has infinitely many critical points.

4 The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = [(0,0]$ for $(x,y) = \dots, (-\pi,0), (0,0), (\pi,0), (2\pi,0), \dots$. These points are angles for which the pendulum is at rest.

5 The function $f(x,y) = |x| + |y|$ is differentiable on the first quadrant. It does not have critical points there. The function has a minimum at $(0,0)$ but it is not in the domain, where the gradient ∇f is defined. The point $(0,0)$ is a minimum but the gradient is not defined there.

In one dimension, the condition $f'(x) = 0, f''(x) > 0$ assured a local minimum, $f'(x) = 0, f''(x) < 0$ for a local maximum. If $f'(x) = 0, f''(x) = 0$, then the situation is not yet clear: a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat inflection point like for $f(x) = x^3$.

Let $f(x,y)$ be a function of two variables with a critical point (a,b). Define $D = f_{xx}f_{yy} - f_{xy}^2$. It is called the **discriminant** of the critical point.

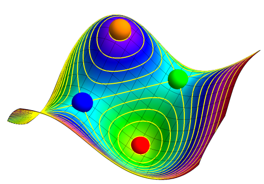
[1]This definition used here does not include points, where f or its derivative is not defined. We usually assume that functions are nice. Points where f is not smooth are or boundary points are considered separately.

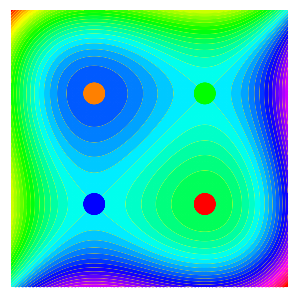
The discriminant can be remembered better if knowing that it is the determinant of the **Hessian** 2×2 **matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

Second derivative test. Assume (a, b) is a critical point for $f(x, y)$.
If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.
If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.
If $D < 0$ then (a, b) is a saddle point.

In the case $D = 0$, we need higher derivatives to determine the nature of the critical point.

6 The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x, y) = [x^2 - 1, -y^2 + 1]$. There are 4 critical points $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1, 1)$ we have $D = -4$ and so a saddle point,
For $(-1, 1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1, -1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1, -1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





To determine the maximum or minimum of $f(x, y)$ on a domain, we find all critical points **in the interior the domain**, then compare their values with maxima or minima **at the boundary**. We will see in the next lecture how to find extrema at the boundary.

7 Find the maximum of $f(x, y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x, -1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and 4/3 is a local maximum on the line $y = -1$. Comparing $f(4/3, 0), f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

Lecture 17: Extrema

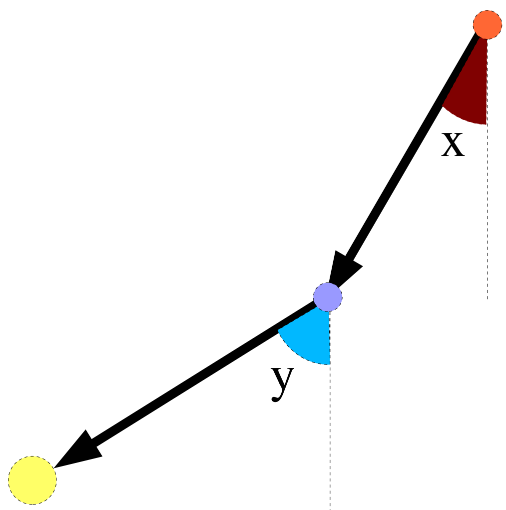
1 Find all the critical points of $f(x,y) = x^2y^2 - x^2 - y^2$ and classify them.

Critical point:	$D = f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	nature
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	

2 The potential energy of a double pendulum is the function

$$f(x, y) = -\cos(x) - \cos(y) \ .$$

At the critical points of this function, the pendulum is at rest. Find them and classify them. Where are the minima, maxima and saddle points?



Critical point:	$D = f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	nature
$(x, y) =$	$D =$	$f_{xx} =$	
$(x, y) =$	$D =$	$f_{xx} =$	
$(x, y) =$	$D =$	$f_{xx} =$	
$(x, y) =$	$D =$	$f_{xx} =$	

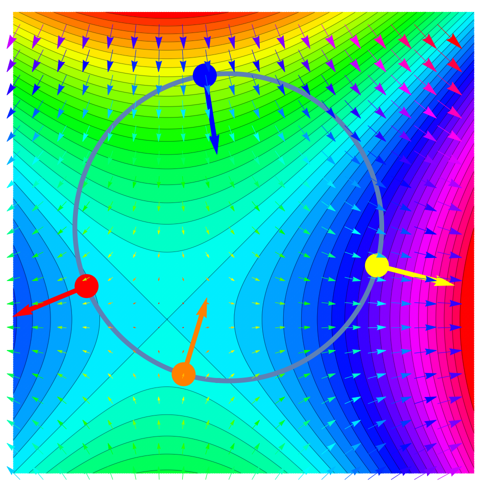
18: Lagrange multipliers

How do we find maxima and minima of a function $f(x,y)$ in the presence of a **constraint** $g(x,y) = c$? A necessary condition for such a "critical point" is that the gradients of f and g are parallel. The reason is that otherwise moving on the level curve $g = c$ will increase or decrease f: the directional derivative of f in the direction tangent to the level curve $g = c$ is zero if and only if the tangent vector to g is perpendicular to the gradient of f or if there is no tangent vector.

The system of equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$ for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is called a **Lagrange multiplier**.

Lagrange theorem: Extrema of $f(x,y)$ on the curve $g(x,y) = c$ are either solutions of the Lagrange equations or critical points of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla g = 0$.



1 Minimize $f(x,y) = x^2 + 2y^2$ under the constraint $g(x,y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

2 Find the shortest distance from the origin $(0,0)$ to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = [2x, 2y], \nabla g = [6x^5, 6y]$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1-x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1,0), (-1,0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

3 Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r,h) = h\pi r^2 = 1$. The surface area is $A(r,h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. Calculate $\nabla f(x,y) = (2y, 2x + 4\pi y), \nabla g(x,y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$.

4 On the curve $g(x,y) = x^2 - y^3$ the function $f(x,y) = x$ obviously has a minimum $(0,0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x,y) = 0$.

Remarks.
1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
2) With $g(x,y) = 0$, the Lagrange equations can also be written as $\nabla F(x,y,\lambda) = 0$ where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$.
3) Either of the two properties equated in the Lagrange theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions. The proof is the same:

Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.

5 Find the extrema of $f(x,y,z) = z$ on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x,y,z) = (0,0,1)$, $\nabla g(x,y,z) = (2x, 2y, 2z)$ and solve $(0,0,1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0,0,-1)$ the maximum the north pole $(0,0,1)$.

6 A dice shows k eyes with probability p_k. Introduce the vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + ... + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The fair dice has maximal entropy. Maximal entropy means **least information content**. An unfair dice provides additional information and allows a cheating gambler or casino to gain profit.

7 Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature minimizes the **free energy** $f(p_1, \dots, p_n) = -\sum_i [p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1-\lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The **Gibbs solution** is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$. [1]

[1] This example appears in a book of Rufus Bowen, Lecture Notes in Math, 470, 1978

Lecture 18: Lagrange

The mathematician and economist Charles W. Cobb (1875-1949) at Amherst college and the economist and politician Paul H. Douglas (born at Salem) also teaching at Amherst found in 1928 empirically a formula $F(L, K) = bL^{\alpha}K^{\beta}$ which gives the total production F of an economic system as a function of the amount of labor L and the capital investment K.

By fitting data, they got $b = 1.01, \alpha = 0.75, \beta = 0.25$. By rescaling the production unit we can get $b = 1$ and work with the formula:

$$F(L, K) = L^{3/4}K^{1/4}$$

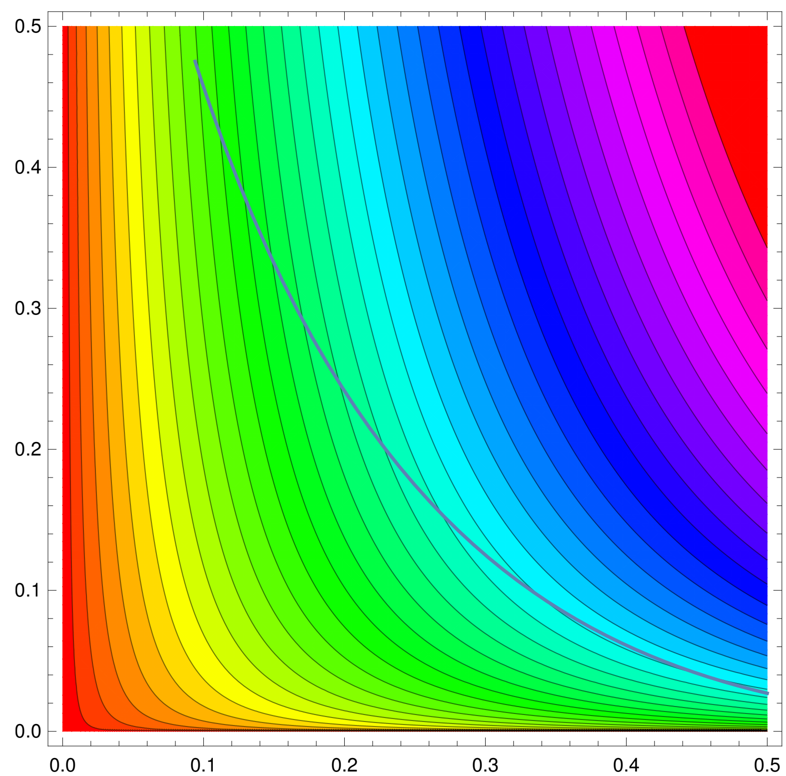
Assume that the labor and capital investment are bound by the constraint $G(L, K) = L^{3/4} + K^{1/4} = 1$. Where is the production P maximal under this constraint?

Paul Douglas, 1892-1976

1 $\nabla F(L, K) =$

2 $\nabla G(L, K) =$

3 Solve: $\nabla F(L, K) = \lambda \nabla G(L, K), G(L, K) = 1$:



19: Global Extrema

To determine the maximum or minimum of $f(x,y)$ on a region, we find first all critical points **in the interior the domain**, then compute all critical points **at the boundary**. This involves to solve an extremal problems with constraints and one without constraints. The largest value among all critical values leads to the maximum.

A point (a,b) is called a **global maximum** of $f(x,y)$ on a region G $f(x,y) \leq f(a,b)$ for all (x,y) in G. If G is not specified, we assume $G = \mathbb{R}^2$. For example, the point $(0,0)$ is a global maximum of the function $f(x,y) = 1 - x^2 - y^2$. Similarly, we call (a,b) a **global minimum**, if $f(x,y) \geq f(a,b)$ for all (x,y).

1 **Question:** Does the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2$ have a global maximum or a global minimum on $\mathbb{R}^2$? If yes, find them. **Solution:** the function has no global maximum on $\mathbb{R}^2$. This can be seen by restricting the function to the x-axis, where $f(x,0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x,y) = (x^2-1)^2 + (y-1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

2 Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on the region $y \geq -1$. **Solution.** With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3,0), f(4/3,-1)$ shows that $(4/3,0)$ is the global maximum.

3 Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them? **Solution.** The critical points satisfy $\nabla f(x,y) = [0,0]$ or $[3x^2 - 3, 3y^2 - 12] = [0,0]$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

There are no global maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

You can ignore the following questions and answers if you like.

1. **Do global extrema always exist?** Yes, if the region Y is **compact** meaning that for every sequence x_n, y_n we can pick a subsequence which converges in Y. This is equivalent that the domain is **closed and bounded**.

Bolzano's extremal value theorem. Every continuous function on a compact domain has a global maximum and a global minimum.

2. **Why are critical points important?** Critical points are relevant in physics because they represent configurations with lowest energy. Many physical laws describe extrema. The Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$ describing a particle of mass m moving in a field V along a path $\gamma : t \mapsto \vec{r}(t)$ are equivalent to the property that the path extremizes the are length $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$ among all paths.

3. **Why is the second derivative test true?** Assume $f(x, y)$ has the critical point $(0, 0)$ and is a quadratic function satisfying $f(0, 0) = 0$. Then $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$ with $A = (x + \frac{b}{a}y), B = b^2/a^2$ and discriminant D. You see that if $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x, y) \neq (0, 0)$. The point $(0, 0)$ is a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x, y) \neq (0, 0)$ and the point (x, y) is a local maximum. If $D < 0$, then f takes both negative and positive values near $(0, 0)$. For a general function approximate by a quadratic one.

4. **Is there something cool about critical points?** Yes, assume $f(x, y)$ be the height of an island. Assume there are only finitely many critical points and all of them have nonzero determinant. Label each critical point with a $+1$ if it is a maximum or minimum, and with -1 if it is a saddle point. The sum of all these numbers is 1, independent of the island. [1]

5) **Can we avoid Lagrange by substitution?** To extremize $f(x, y)$ under the constraint $g(x, y) = 0$ we find $y = y(x)$ from the second equation and extremize the single variable problem $f(x, y(x))$. To extremize $f(x, y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution.

6) **Is there a second derivative test for Lagrange?** A second derivative test can be designed using second directional derivative in the direction of the tangent. Instead, we just make a list of critical points and pick the maximum and minimum.

7) **Does Lagrange also work with more constraints?** With two constraints the constraint $g = c, h = d$ defines a curve. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f. Here is a formulation in three dimensions. Extrema of $f(x, y, z)$ under the constraint $g(x, y, z) = c, h(x, y, z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x, y, z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

8) **Why do D and f_{xx} appear in the second derivative test**. They are natural. The discriminant D is a determinant $\det(H)$ of the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. If $D > 0$ then the sign of f_{xx} is the same as the sign of the trace $f_{xx} + f_{yy}$ which is coordinate independent too. The determinant is the product $\lambda_1 \lambda_2$ of the eigenvalues of H and the trace is the sum of the eigenvalues.

9) **What does D mean?** The discriminant D is defined also at points where we have no critical point. The number $K = D/(1 + |\nabla f|^2)^2$ is called the **Gaussian curvature** of the surface. At critical points $K = D$. Curvature is remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. [2]

10) **Is there a 2. derivative test in higher dimensions?** Yes. one can form the second derivative matrix H and look at all the eigenvalues of H. If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. In general eigenvalues have different signs and we have a saddle point type.

[1] This follows from the **Poincare-Hopf** theorem.

[2] This is the **Theorema Egregia of Gauss**.

Lecture 19: Global extrema

The task to find global maxima or global minima can be quite a bit of work. We have to do two things. First find the critical points in the interior, then find the critical points at the boundary using the Lagrange multiplier method.

1 Find the local maxima and minima of the function $f(x,y) = x^2+y^4+x-2y^2$ in the interior of the region $x^2+y^2 \leq 4$. Classify them using the second derivative test.

This is a bit of review of the Monday lecture on extrema. There will be three points.

Critical point:	$D = f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	nature
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	
$(x,y) =$	$D =$	$f_{xx} =$	

2 Find the local maxima and minima of the same function $f(x,y) = x^2+y^4+x+2y^2$ on the boundary of the region $x^2+y^2 \leq 4$. Hint: there will be 8 points.

3 Compare the list of all critical points, the ones in the interior and the ones on the boundary to find the maximum and minimum.

4 Now forget about the region we have considered initial and look at the function $f(x, y)$ on the entire plane $\mathbb{R}^2$. Does the function $f(x, y)$ have a global maximum on $\mathbb{R}^2$?

5 Does the function $f(x, y)$ have a global minimum on the entire plane $\mathbb{R}^2$?

1

[1] $(0,-1), (0,0), (0,1)$ are the critical points and $(0,-2), (0,2), (2,0), (-2,0), (\pm\sqrt{7/2}, \pm\sqrt{1/2})$ are the critical points on the boundary.

Lecture 20: Double integrals

The integral $\int\int_R f(x,y)\ dxdy$ is defined as the limit of Riemann sums

$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

when $n \to \infty$.

1 In order to integrate $f(x,y) = xy$ over the unit square, we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Integrating with respect to y from 0 to 1 gives $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\iint_R f(x,y)\ dxdy$ as the **signed volume** of the solid below the graph of f and above the region R in the $x-y$ plane. The volume below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle, if the function f is continuous: $\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$.

Proof. For every n, check the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

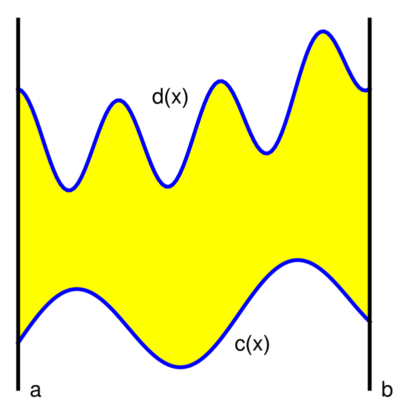
holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.

A **dy dx region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \}\ .$$

An integral over such a region is called a **dy dx integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx\ .$$

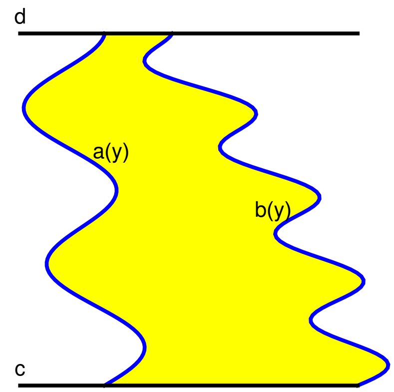


A **dx dy region** is of the form

$$R = \{(x,y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **dx dy integral**

$$\int\int_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



4 Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

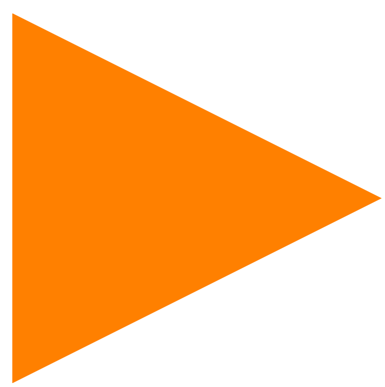
$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

This can be solved by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} .$$

5 Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a dy dx integral. Because we would have to compute 2 different integrals as a dy dx integral. The y bounds are $x = y-1$ and $x = 1-y$

$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4}-\frac{1}{3}) = \frac{1}{10} .$$



6 Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

$$\int\int_R e^{-x^2}\ dxdy\ ?$$

The dx dy integral $\int_0^1[\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions. The dy dx integral $\int_0^1[\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$

Lecture 20: Double integrals

1 Find the area of the region enclosed by the curves $y = x - x^3$ and $y = x/4$.

2 Set up a double integral of a function $f(x, y)$ over the region bound by $y = 2x$ and $x = 2y^2$ and $y = 1$.

3 Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x,y)\,dxdy$
	$\int_0^1 \int_y^1 f(x,y)\,dxdy$
	$\int_0^1 \int_0^{1-y} f(x,y)\,dxdy$
	$\int_0^1 \int_0^y f(x,y)\,dxdy$

I) $\int_0^1 \int_0^x f(x,y)\,dydx$

II) $\int_0^1 \int_0^{1-x} f(x,y)\,dydx$

III) $\int_0^1 \int_x^1 f(x,y)\,dydx$

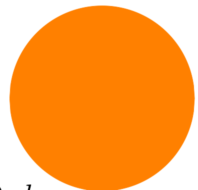
IV) $\int_0^1 \int_0^{x-1} f(x,y)\,dydx$

V) $\int_0^1 \int_{1-x}^1 f(x,y)\,dydx$

Lecture 21: Polar integration

1 The area of a disc of radius R is

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \; dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2} \; dx \; .$$



This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)} R\cos(u) \; du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u) \; du \; .$$

Using a double angle formula we get $R^2 \int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u)}{2} \; du = R^2\pi$. We will now see how to do that better in polar coordinates.

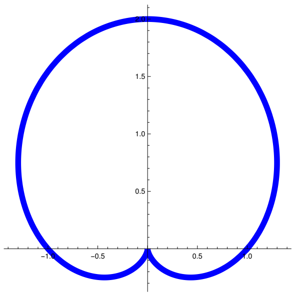
A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

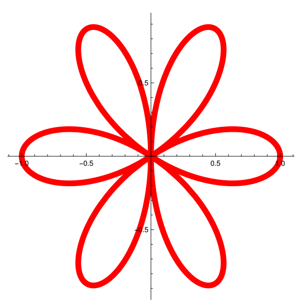
In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = [r(t)\cos(\theta(t), r(t)\sin(\theta(t))]$. We are especially interested in regions which are bound by **polar graphs**, where $\theta(t) = t$.

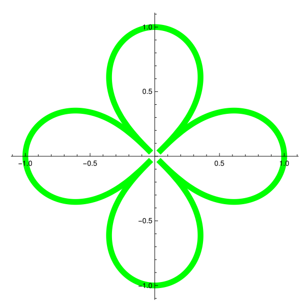
2 The **polar region** defined by $r \leq |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$ they are also called **rhododenea**. These names reflect that polar regions model flowers well.

3 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of a **limacon** a polar curve of the form $r(\theta) = 1 + b\sin(\theta)$.

4 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign. It encloses a flower with two petals.







To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y) \; dxdy = \iint_R f(r\cos(\theta), r\sin(\theta) r \; drd\theta$$

5 Integrate

$$f(x,y) = x^2 + x^2 + xy ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

6 We have earlier computed area of the disc $\{x^2 + y^2 \leq R^2\ \}$ using substitution. It is more elegant to do this integral in polar coordinates: $\int_0^{2\pi} \int_0^R r\ drd\theta = 2\pi r^2/2|_0^R = \pi R^2$.

Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\ d\theta dr$.

7 Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2\ .$$

8 Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

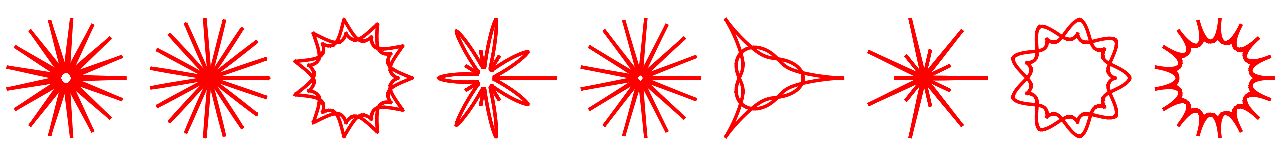
$$\int_1^2 \int_0^{\pi} r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\ d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

9 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



[1]"Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

Lecture 21: Polar integration

1 Evaluate the integral

$$\int \int_R \frac{1}{\sqrt{x^2+y^2}} \, dxdy$$

over the disc of radius 1.

2 The integral

$$2 \int_0^{\pi/2} \int_{\theta}^{2\theta} r \; drd\theta$$

computes the area of the region shown below. Can you see why?

3 Find the area.

4 Integrate $f(x,y) = x^2$ over the unit disk $\{2 \leq x^2+y^2 \leq 9\}$.

5 Can you evaluate the following integral? (Halloween problem!)

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \; drd\theta \; .$$

Hint: Write it in more convenient coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dydx \; .$$

This is a quarter disc in the x, y plane. Now use polar coordinates.

Lecture 22: Surface area

A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u,u+du] \times [v,v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r}+\vec{r}_u]$ and $[\vec{r}, \vec{r}+\vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv$.

1 The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

For a planar region $\vec{r}(s,t) = P + sv + tw$ where $(s,t) \in G$, the surface area is the area of G times $|v \times w|$.

2 The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]$ maps the rectangle $G = [0,2\pi] \times [0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \; dS = \int_0^{2\pi}\int_0^{\pi} L^2 \sin(v) \; dvdu = 4\pi L^2$

For a sphere of radius L, we have $|\vec{r}_u \times \vec{r}_v| = L^2\sin(v)$ The surface area is $4\pi L^2$.

3 For graphs $(u,v) \mapsto [u,v,f(u,v)]$, we have $\vec{r}_u = (1,0,f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1+f_u^2+f_v^2} \; dudv$.

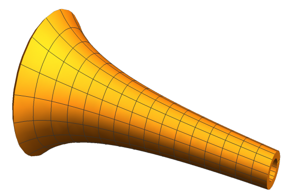
For a graph $z = f(x,y)$ parametrized over G, the surface area is

$$\int\int_G \sqrt{1+f_x^2+f_y^2} \; dxdy \; .$$

4 Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]$ on $R = [0,2\pi] \times [a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \; dudv = 2\pi \int_a^b |f(v)|\sqrt{1+f'(v)^2} \; dv$.

For a surface of revolution $r = f(z)$ with $a \leq z \leq b$, the surface area is

$$2\pi \int_a^b |f(z)| \sqrt{1 + f'(z)^2} \, dz \; .$$



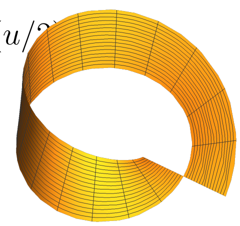
5 Gabriel's trumpet is the surface of revolution where $g(z) = 1/z$, where $1 \leq z < \infty$. Its volume is $\int_1^\infty \pi g(z)^2 \; dz = \pi$. We will compute in class the surface area.

6 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 9$. **Solution.** We use polar coordinates in the yz-plane. The paraboloid is parametrized by $(u, v) \mapsto (v^2, v\cos(u), v\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \; dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2} \; dudv = 2\pi \int_0^3 v\sqrt{1+4v^2} \; dv = \pi(37^{3/2} - 1)/6$.

7 In this example we derive the distortion factor r in polar coordinates. To do so, we parametrize a region in the xy plane with $\vec{r}(u, v) = [u\cos(v), u\sin(v), 0]$. Given a region G in the uv plane like the rectangle $[0, \pi] \times [1, 2]$, we obtain a region S in the xy plane as the image. The factor $|\vec{r}_u \times \vec{r}_v|$ is equal to the radius u. In our example, the surface area is $\int_0^\pi \int_1^2 u \; dudv = \pi(4 - 1) = 3\pi$. This is the area of the half annulus S. We could have used polar coordinates directly in the xy plane and compute $\int_0^\pi \int_1^2 r \; dr \; d\theta = 3\pi$. But the only thing which has changed are the names of the variables.

The surface parametrized by

$$\vec{r}(u, v) = [(2+v\cos(u/2))\cos(u), (2+v\cos(u/2))\sin(u), v\sin(u/2)]$$



8 on $G = [0, 2\pi] \times [-1, 1]$ is called a **Möbius strip**. What is its surface area? **Solution.** The calculation of $|\vec{r}_u \times \vec{r}_v|^2 = 4 + 3v^2/4 + 4v\cos(u/2) + v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0, 2\pi] \times [-1, 1]$ can only be evaluated numerically, the result is 25.413....

Lecture 22: Surface area

For a parametric surface, the surface area is defined as

$$\int\int_R |\vec{r}_u \times \vec{r}_v| \cdot dudv \; .$$

For a **surface of revolution** parameterized by

$$\vec{r}(\theta, z) = [g(z)\cos(\theta), g(z)\sin(\theta)] \; .$$

we get

$$|\vec{r}_\theta \times \vec{r}_z| = |g(z)|\sqrt{1 + g'(z)^2} \; .$$

The surface area of such a surface of revolution is

$$2\pi \int_a^b |g(z)|\sqrt{1 + g'(z)^2} \, dz \; .$$

1 Find the surface area of the surface

$$\vec{r}(u, v) = [u, v, 2u]$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 3$.

2 Find the surface area of the surface

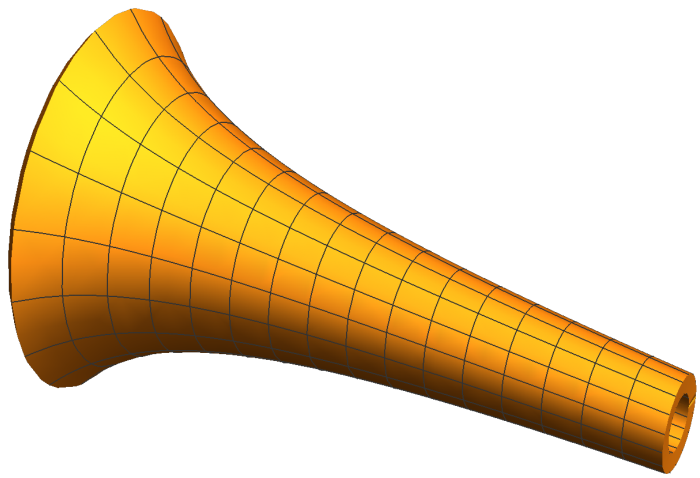
$$\vec{r}(u, v) = [u, v, u^2]$$

where $0 \leq u \leq 1$ and $0 \leq v \leq u$.

Gabriel's trumpet is the surface of revolution where $g(z) = 1/z$, where $1 \leq z < \infty$.

3 Verify that the volume of the trumpet is $\int_1^\infty \pi g(z)^2 \, dz = \pi$.

4 Compute the surface area integral of the trumpet.



We conclude that the trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

Lecture 23: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n) \in E} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \ .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 Assume E is the box $[0,1] \times [0,1] \times [0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx = 1$. When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

The two important methods for triple integrals are the "washer method" and the "sandwich method". The washer method from single variable calculus reduces the problem directly to a one dimensional integral. The new sandwich method reduces the problem to a two dimensional integration problem.

The **washer method** slices the solid along the z-axes. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b [\int\int_{R(z)} f(x,y,z)\ dxdy]\ dz$. The **sandwich method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral becomes $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz]\ dxdy$.

2 An important special case of the sandwich method is the volume

$$\int\int_R \int_0^{f(x,y)} 1\ dzdxdy \ .$$

under the graph of a function $f(x,y)$ and above a region R. It is the integral $\int\int_R f(x,y)\ dA$. What we actually have computed is a triple integral

3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

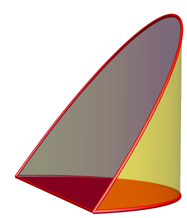
$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA .$$

which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2}\, dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2} r \, dr d\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2)\, dz = 4\pi/3$.

4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z)\, dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1. **Solution:**

$$\int_0^2 \int_0^6 \int_0^{4-x^2} \, dz\, dy\, dx = \int_0^2 \int_0^6 (4 - x^2)\, dydx$$
$$= 6 \int_0^2 (4 - x^2)\, dx = 6(4x - x^3/3)|_0^2 = 32$$

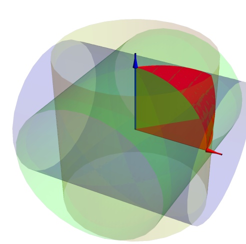
5 The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution**. Look from the situation from above and picture it in the $x-y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\ dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2}\int_0^1 r^2 \cos(\theta)\, dr d\theta = 2/3$.



6 Finding the volume of the solid region bound by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2\cos^2(\theta)} r\, dr d\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x) + \sec(x) + \tan(x)$. The result is $16 - 8\sqrt{2}$.



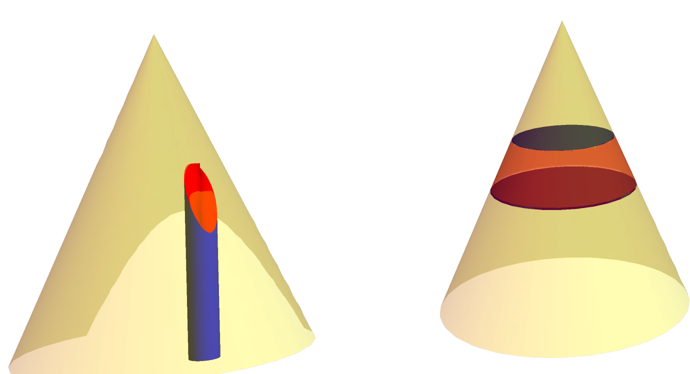
Lecture 23: Triple integrals

1 Evaluate the integral

$$\int_0^{2\pi} \int_u^{2\pi} \int_0^{\sqrt{1+v^2}} 5 \, dzdvdu \, .$$

2 Integrate $f(x, y, z) = xz$ over the hoof solid $x^2 + y^2 \leq 1, 0 \leq z \leq x$. The hoof solid was considered by Archimedes already.

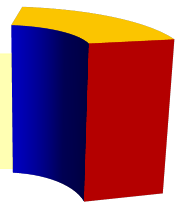
3 a) Find the volume of the cone with base radius 2 and height 2 by reducing it to a single variable integral.
b) Find the volume of a cone with base radius 2 and height 2 by reducing it to a double integral.



Lecture 24: Spherical integration

Cylindrical coordinates are coordinates in space in which polar coordinates are chosen in the xy-plane and where the z-coordinate is left untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, produces the same integration factor $\boxed{r}$ as in polar coordinates.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(r,\theta,z)\ \boxed{\text{r}}\ drd\theta dz$$

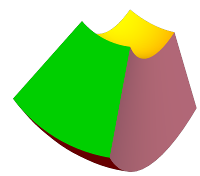


In spherical coordinates we use the distance ρ to the origin as well as the polar angle θ as well as ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

It produces an integration factor is the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(\rho,\theta,\phi)\ \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d\phi$$



1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\ d\phi d\theta d\rho \ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.

The moment of inertia of a body G with respect to an z axes is defined as the triple integral $\int\int\int_G x^2 + y^2\ dzdydx$, where r is the distance from the axes.

For a sphere of radius R we obtain with respect to the z-axis:

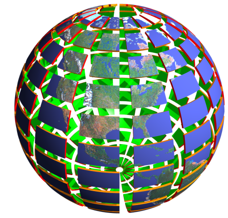
2

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho$$

$$= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta)$$

$$= (\int_0^\pi \sin(\phi)(1-\cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta)$$

$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} \ .$$

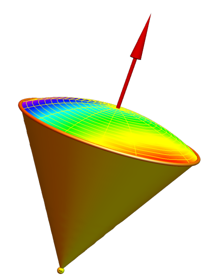


If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi) \; d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

4 Find $\int\int\int_R z^2 \; dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4 \}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have



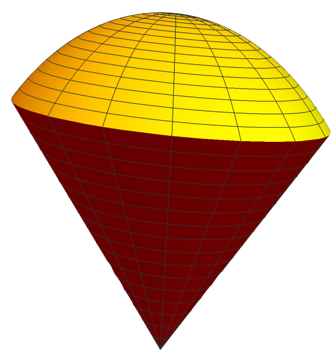
$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi) \; d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2}) \; .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.

Lecture 24: Spherical integration

1 Find the volume of $\int\int\int_E 1 \; dxdydz$ of the solid E obtained by intersecting the sphere $x^2+y^2+z^2 \leq 4$ with the one sided cone $x^2+y^2 \leq z^2, z \geq 0$.



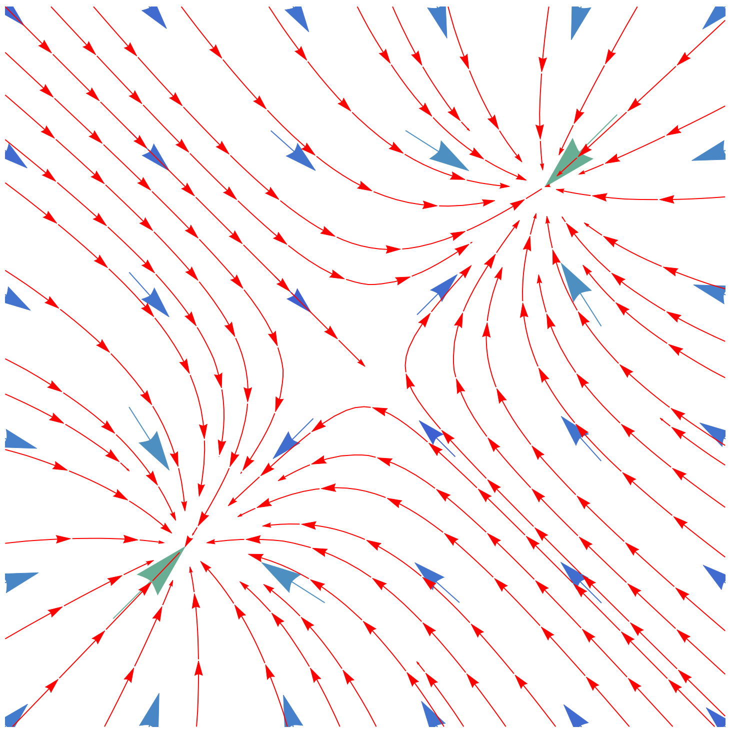
2 Write down the integral for the moment of inertia $\int\int\int_E x^2 + y^2 \; dxdydz$ of this solid E.

3 Set up the integral for $\int\int\int_E z^2 \; dxdydz$ for the solid which is the intersection of $x^2+y^2+z^2 \leq 4$ and $x^2+y^2 \geq z^2$ and $x > 0$.

Lecture 25: Vector fields

A **vector field** in the plane is a map, which assigns to each point (x, y) a vector $\vec{F}(x, y) = [P(x, y), Q(x, y)]$. A vector field in space is a map, which assigns to (x, y, z) in space a vector $\vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]$.

For example, $\vec{F}(x, y) = [x - 1, y]/((x - 1)^2 + y^2)^{3/2} - [x + 1, y]/((x + 1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called the **dipole field**. It is shown in the picture below:



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? If $\vec{F}(x, y) = [P(x, y), Q(x, y)] = \nabla f(x, y) = [f_x(x, y), f_y(x, y)]$ then $Q_x(x, y) = P_y(x, y)$ by Clairaut. If this does not hold at some point, then F is no gradient field.

Clairaut test: if $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ is not a gradient field.

We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for $\vec{F}$ to be a gradient field. In class, we see more examples on how to construct the potential f from the gradient field F.

1 Is the vector field $\vec{F}(x,y) = [P,Q] = [3x^2y + y + 2, x^3 + x - 1]$ a gradient field? **Solution**: the Clairaut test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x,y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^2 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

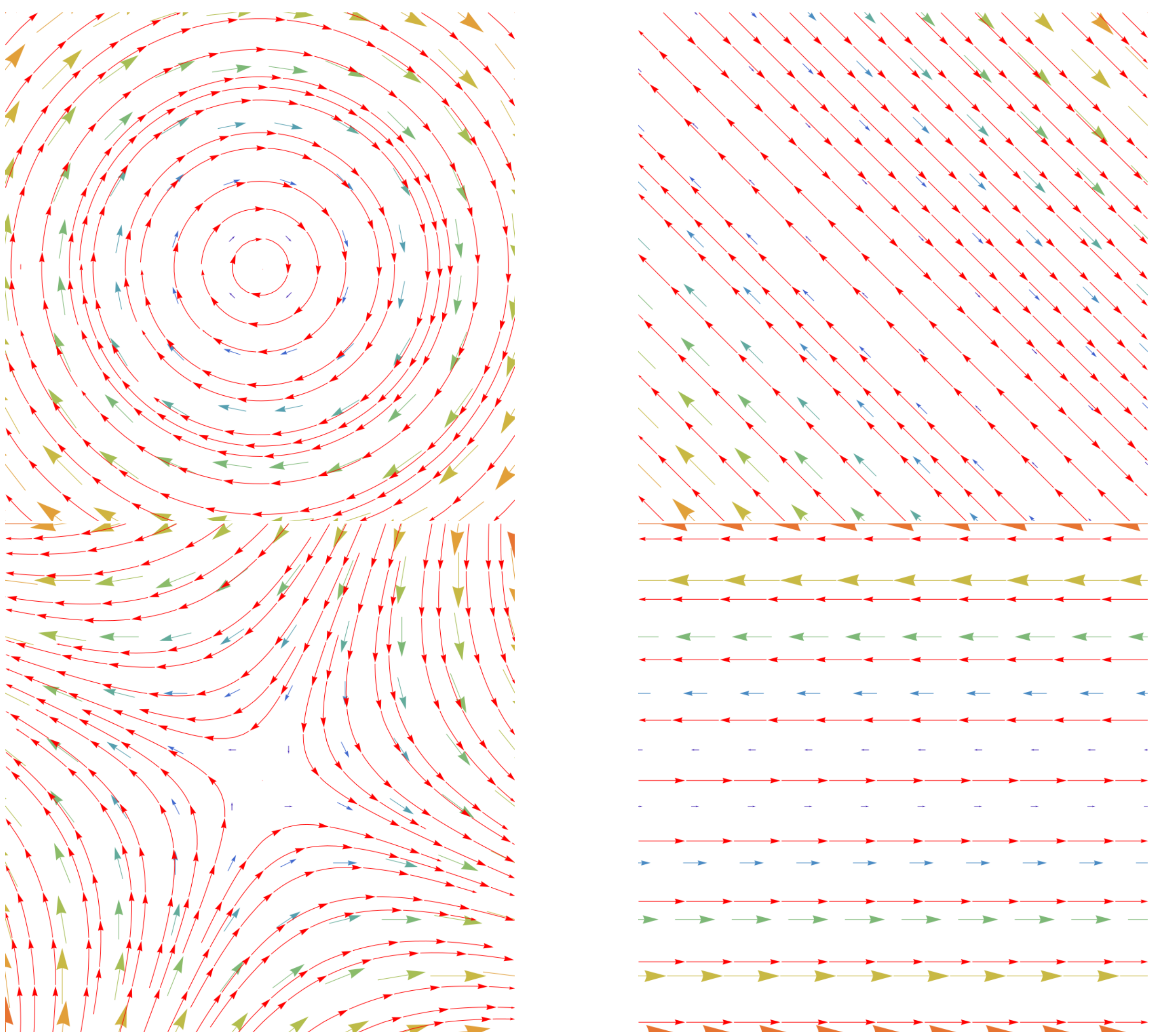
2 Is the vector field $\vec{F}(x,y) = [xy, 2xy^2]$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

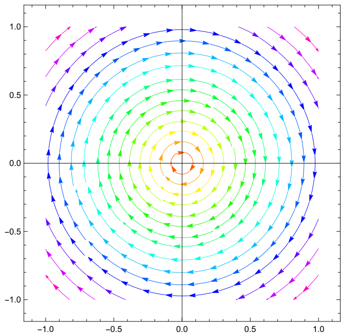
Vector fields are important in differential equations:

3 A class of vector fields important in mechanics are **Hamiltonian fields**: If $H(x,y)$ is a function of two variables, then $[H_y(x,y), -H_x(x,y)]$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x,y) = x^2 + y^2$. Its vector field $(H_y(x,y), -H_x(x,y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H.

4 Is the vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)] = [xy, x^2]$ a gradient field?
No. Is the vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)] = [\sin(x) + y, \cos(y) + x]$ a gradient field?
Yes. the function is $f(x,y) = -\cos(x) + \sin(y) + xy$.

5 Can you spot the following vector fields in the pictures? $F(x,y) = [-y, 0]$, $F(x,y) = [-y - x, x + y]$, $F(x,y) = [y, -x]$, $F(x,y) = [y - x, x + y]$. Which ones are conservative?





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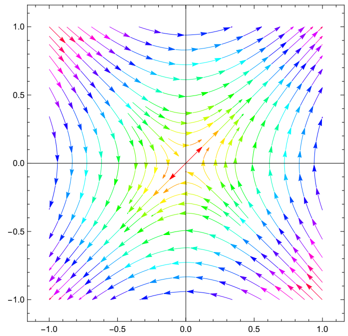

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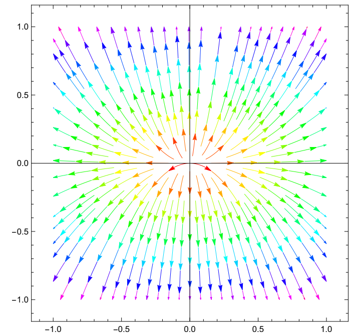

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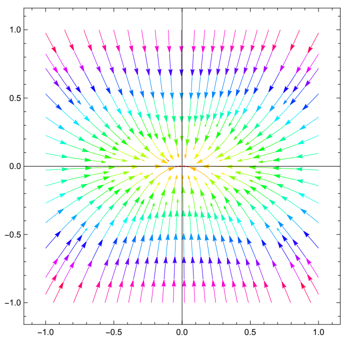

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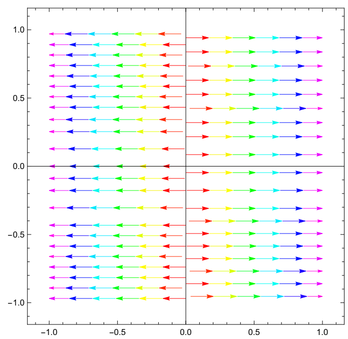

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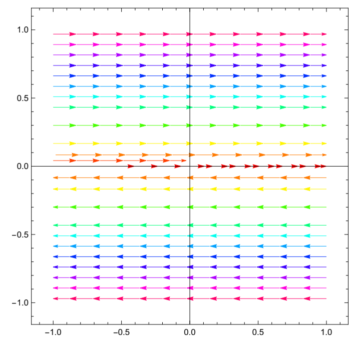

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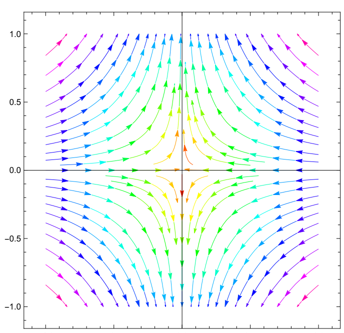

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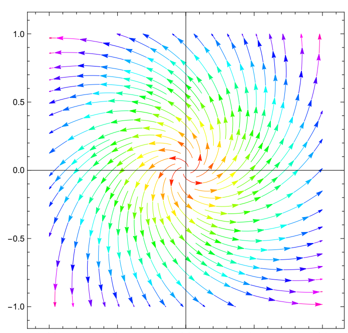

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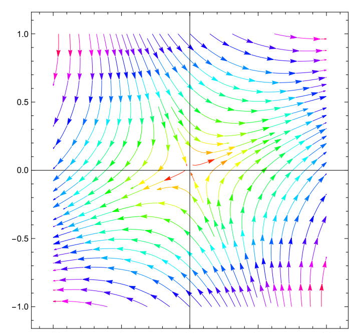

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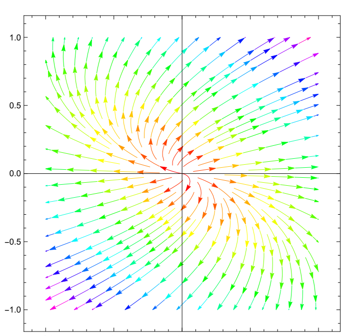

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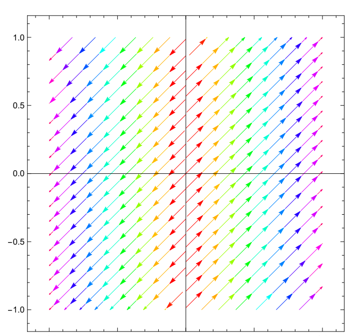

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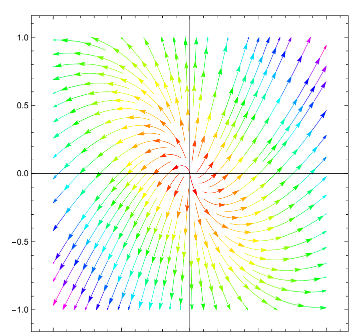


12

Match the vector fields with the corresponding phase portraits.

Vector field	1	2	3	4	5	6	7	8	9	10	11	12
$\vec{F}(x,y) = [y, -x]$												
$\vec{F}(x,y) = [x, 2y]$												
$\vec{F}(x,y) = [y, x]$												
$\vec{F}(x,y) = [x, 0]$												
$\vec{F}(x,y) = [-x, -2y]$												
$\vec{F}(x,y) = [-x, y]$												
$\vec{F}(x,y) = [y, 0]$												
$\vec{F}(x,y) = [x+y, x-y]$												
$\vec{F}(x,y) = [x-y, x+y]$												
$\vec{F}(x,y) = [x, x]$												
$\vec{F}(x,y) = [x+y, y]$												
$\vec{F}(x,y) = [x, x+y]$												

Lecture 26: Line integrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The short-hand notation $\int_C \vec{F} \cdot d\vec{r}$ is also used. In physics, if $\vec{F}(x, y, z)$ is a force field, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ is called **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ gives the **electric potential**.

The line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the parametrization of the curve but unlike arc length, it depends in which direction we go over the curve.

1 Let $C : t \mapsto \vec{r}(t) = [\cos(t), \sin(t)]$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = [-y, x]$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.
Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} [-\sin(t), \cos(t)] \cdot [-\sin(t), \cos(t)]\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$

2 Let $\vec{r}(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = [-xy, 0]$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$. **Solution:** In Cartesian coordinates, the curve is $\vec{r}(t) = [\cos^2(t), \cos(t)\sin(t)]$. The velocity vector is then $\vec{r}\,'(t) = [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t)]$. The line integral is

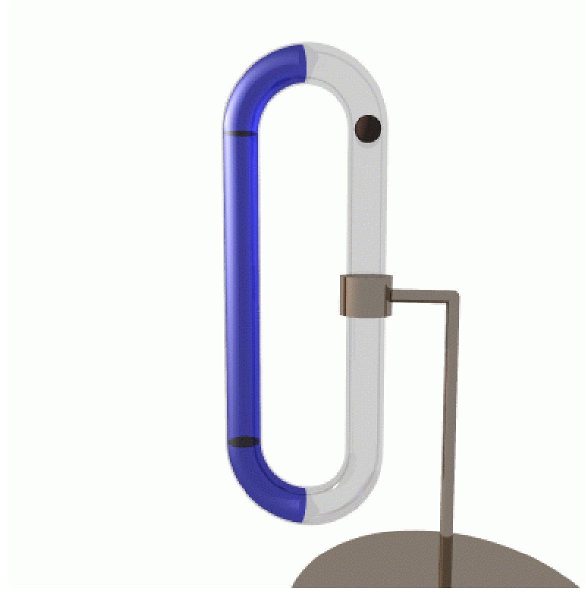
$$\int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{\pi} [\cos^3(t)\sin(t), 0] \cdot [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)]\, dt$$

$$= -2\int_0^{\pi} \sin^2(t)\cos^4(t)\, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^{\pi} = -\pi/8\,.$$

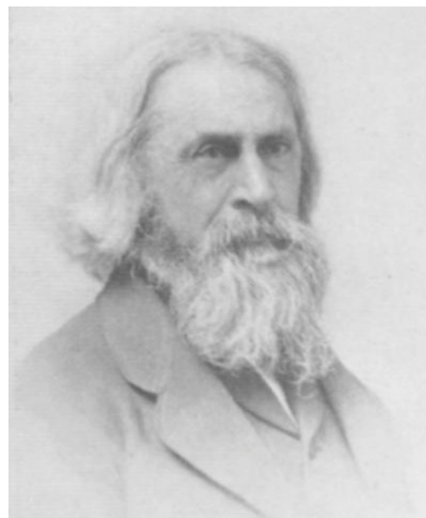
3 Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

Something to think about:

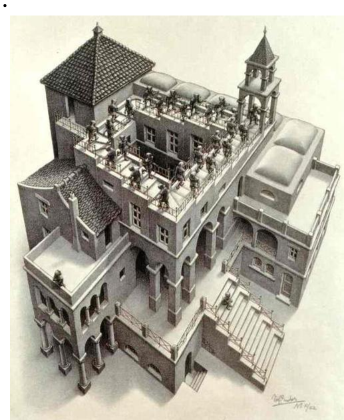
A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which along some closed loops the energy gain is nonnegative. By possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.

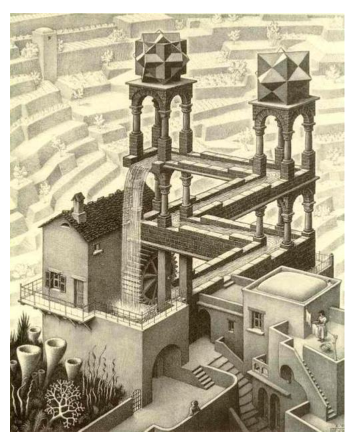


Why does this "perpetual motion machine" not work? The former Harvard professor Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the "antropic principle". "Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".



Nonconservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?





Lecture 26: Line integrals

1 What is the line integral of

$$\vec{F}(x,y) = [x^2 + y, x - y]$$

along the ellipse $x^2 + y^2/4 = 1$ parametrized counter clockwise.

2 Find the line integral of a force field

$$\vec{F} = [-y, -4, 1]$$

along the path $\vec{r}(t) = [t, 3t, t]$ from $t = 0$ to $t = 1$.

3 Can you see intuitively, why the line integral along a closed curve is zero, if $\vec{F}$ is a constant vector field.

4 For which closed curves is the line integral along the vector field $\vec{F} = [-y, x]$ positive?

Lecture 27: Fundamental theorem of line integrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The following theorem generalizes the fundamental theorem of calculus to higher dimensions:

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a))\ .$$

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a))\ .$$

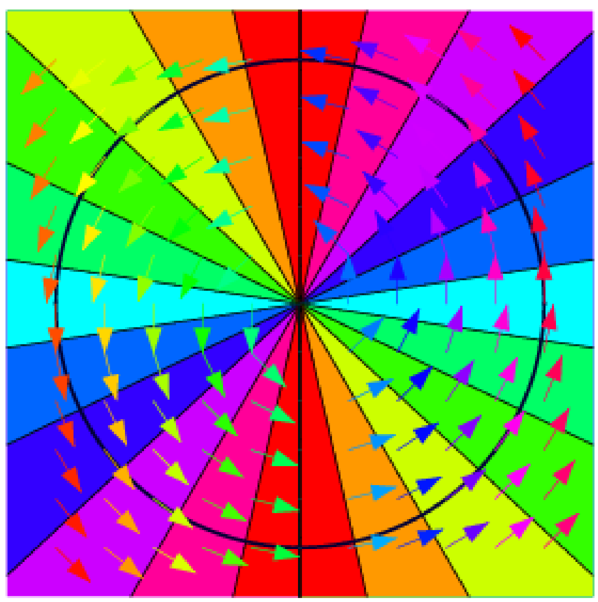
1 Let $\vec{F}(x, y) = [2xy^2 + 3x^2, 2yx^2]$ be the force field produced by the water near the rheinfalls in Switzerland. Find the line integral along a line from $(0, 0)$ to $(2, 1)$. Solution. We find a potential $f(x, y) = x^2y^2 + x^3$ and instead compute the difference of the potential values which is 12.

2 Let $\vec{F}(x,y) = [2xy^2 + 3x^2, 2yx^2]$. Find a potential f of $\vec{F} = [P,Q]$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g, h from $f(x,y) = \int_0^x P(t,y)\, dt + h(y)$ and $f_y(x,y) = g(x,y)$.

3 Here is an enigma. Let $\vec{F}(x,y) = [P,Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P$, $f_y = (1/x)/(1+y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\, \vec{dr}$, where γ is the unit circle is

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}] \cdot [-\sin(t), \cos(t)]\, dt$$

which is $\int_0^{2\pi} 1\, dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined. The region in which $\vec{F}$ is defined is not simply connected.



Lecture 27: Fundamental theorem of line integrals

1 What is the line integral of

$$\vec{F}(x,y) = [\cos^{2019}(x+y), \cos^{2019}(x+y)+x]$$

along the unit circle parametrized counter clockwise.

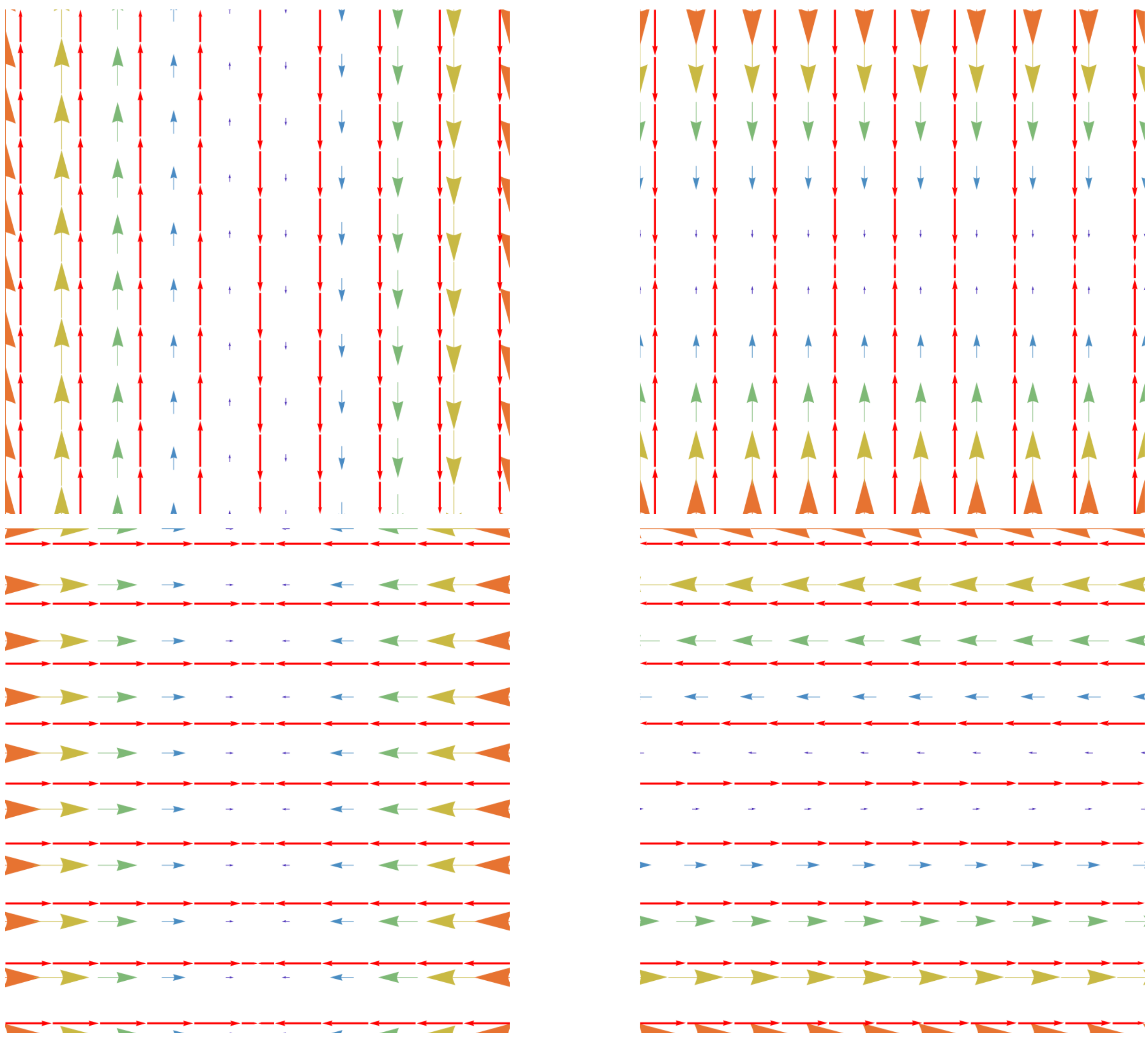
2 Find the line integral of

$$\vec{F} = [\sin(x) - y, y^7 + 5]$$

along the path $\vec{r}(t) = [t, \sin(2t)]$ from $t = 0$ to $t = \pi$.

Remember that a vector field is a **gradient field** if $\vec{F} = \nabla f$. There are equivalent notions which are all called **conservative** F is a **gradient field**, F has the **closed loop property** or F is **path independent**. If a vector field is defined everywhere in the plane, then $\text{curl}(F) = Q_x - P_y = 0$ is equivalent too as we will see later.

3 Quantitatively decide which of the following vector fields are conservative.



Lecture 28: Green's theorem

The **curl** of a vector field $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ is defined as the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y) .$$

The function curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = [P, Q]$ is the scalar $Q_x - P_y$.

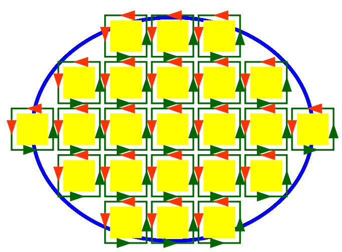
1 For $\vec{F}(x, y) = [-y, x]$ we have $\text{curl}(F)(x, y) = 2$.

2 If $\vec{F}(x, y) = \nabla f$ is a gradient field then the curl is zero because if $P(x, y) = f_x(x, y), Q(x, y) = f_y(x, y)$ and $\text{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut's theorem.

Green's theorem: If $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ is a vector field and R is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \int \int_G \text{curl}(F) \; dxdy .$$

Proof. The integral of $\vec{F}$ along the boundary of $G = [x, x+\epsilon] \times [y, y+\epsilon]$ is $\int_0^\epsilon P(x+t, y)dt + \int_0^\epsilon Q(x+\epsilon, y+t) \; dt - \int_0^\epsilon P(x+t, y+\epsilon) \; dt - \int_0^\epsilon Q(x, y+t) \; dt$. Because $Q(x+\epsilon, y) - Q(x, y) \sim Q_x(x, y)\epsilon$ and $P(x, y+\epsilon) - P(x, y) \sim P_y(x, y)\epsilon$, this is is $(Q_x - P_y)\epsilon^2 \sim \int_0^\epsilon \int_0^\epsilon \text{curl}(F) \; dxdy$. All identities hold in the limit $\epsilon \to 0$.



A general region G can be cut into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vortex strength $Q_x - P_y$ on the squares is a Riemann sum approximation of the double integral. The boundary integrals converge to the line integral of C.

George Green lived from 1793 to 1841. He was a physicist a self-taught mathematician and miller.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals and $\int \int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

The already established the Clairaut identity

$\text{curl}(\text{grad}(f)) = 0$

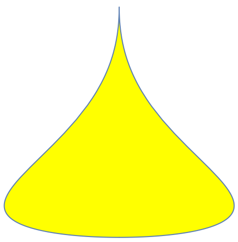
It can also remembered as $\nabla \times \nabla f$ noting that the cross product of two identical vectors is 0. Treating ∇ as a vector is **nabla calculus**.

4 Find the line integral of $\vec{F}(x, y) = [x^2 - y^2, 2xy] = [P, Q]$ along the boundary of the rectangle $[0, 2] \times [0, 1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y + 2y = 4y$ so that $\int_C \vec{F}\ d\vec{r} = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

5 Find the area of the region enclosed by

$$\vec{r}(t) = [\frac{\sin(\pi t)^2}{t}, t^2 - 1]$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = [0, x]$.



6 An important application of Green is to **compute area**. With the vector fields $\vec{F}(x, y) = [P, Q] = [-y, 0]$ or $\vec{F}(x, y) = [0, x]$ have vorticity $\text{curl}(\vec{F})(x, y) = 1$. For $\vec{F}(x, y) = [0, x]$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C [0, x(t)] \cdot [x'(t), y'(t)]\ dt\ .$$

7 Let G be the region under the graph of a function $f(x)$ on $[a, b]$. The line integral around the boundary of G is 0 from $(a, 0)$ to $(b, 0)$ because $\vec{F}(x, y) = [0, 0]$ there. The line integral is also zero from $(b, 0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a, 0)$ because $N = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b [-y(t), 0] \cdot [1, f'(t)]\ dt = \int_a^b f(t)\ dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can be pulled together to a point in R.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

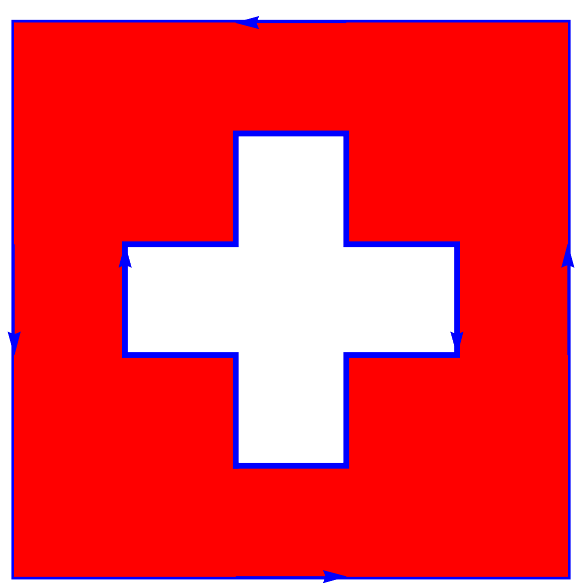
Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\iint_R \text{curl}(\vec{F}))(x, y)\ dx$ $\int_C \vec{F}\ d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G, line integrals are path independent and $\vec{F}$ is a gradient field.

Lecture 28: Greens theorem

1 Let $\vec{F}(x,y) = [x^2 - y, y^2 + x]$ and let $\vec{r}(t)$ be the boundary of the square $0 \leq x \leq 1, 0 \leq y \leq 1$, parametrized counter clockwise.

2 In class we computed the area of the ellipse using $\vec{F}(x,y) = [0, -x]$ and $\vec{r}(t) = [a\cos(t), b\sin(t)]$. It goes faster with $\vec{F}(x,y) = [y, -x]/2$, a field which also has curl constant 1. Do it!

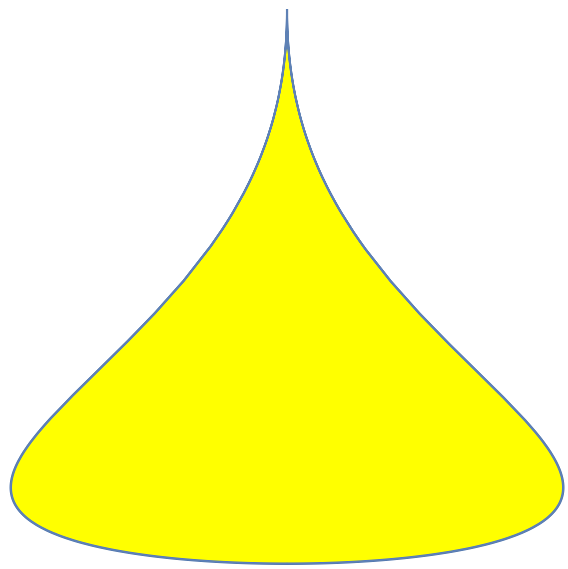
3 Let G be the red complement of the cross in the Swiss flag. The entire flag has dimension 5×5 and the cross consists of 5 squares of unit length. Let C be the boundary of the red complement region oriented so that the region is to the left. The boundary consists of two curves. Find the line integral $\int_C \vec{F}\ \vec{dr}$ for $\vec{F}(x,y) = [x^9 y^{10} + y, y^9 x^{10} - x]$.



4 Find the area of the region enclosed by

$$\vec{r}(t) = [\frac{\sin(\pi t)^2}{t}, t^2 - 1]$$

for $-1 \leq t \leq 1$. Use Greens theorem with $\vec{F} = [0, x]$.



Remarks.

- This problem could not be solved without integral theorem.
- Also $\vec{F} = [-y, 0]$ has curl 1 but the integral would not work.

Lecture 29: Curl, Divergence and Flux

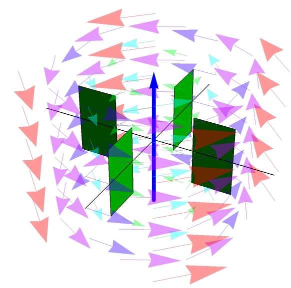
The **curl** of $\vec{F} = [P, Q]$ is $Q_x - P_y$, a scalar field. The **curl** of $\vec{F} = [P, Q, R]$ is

$$\text{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y] .$$

We can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Fields of zero curl are called **irrotational**.

1 The curl of the vector field $[x^2 + y^5, z^2, x^2 + z^2]$ is $[-2z, -2x, -5y^4]$.

If you place a "paddle wheel" pointing into the direction v, its rotation speed $\vec{F} \cdot \vec{v}$. The direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. The angular velocity is the magnitude of the curl.



The **divergence** of $\vec{F} = [P, Q, R]$ is $\text{div}([P, Q, R]) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$. The **divergence** of $\vec{F} = [P, Q]$ is $\text{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

The divergence measures the "expansion" of a field. Fields of zero divergence are **incompressible**. With $\nabla = [\partial_x, \partial_y, \partial_z]$, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$.

$$\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz} .$$

is the Laplacian of f. One also writes $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f)$.

From $\nabla \cdot \nabla \times \vec{F} = 0$ and $\nabla \times \nabla \vec{F} = \vec{0}$, we get

$\text{div}(\text{curl}(\vec{F})) = 0$, $\text{curl}(\text{grad}(f)) = \vec{0}$.

2 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = [x + y, z, y^2] = \text{curl}(\vec{G})$?
Answer: No, because $\text{div}(\vec{F}) = 1$ is incompatible with $\text{div}(\text{curl}(\vec{G})) = 0$.

3 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \text{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \text{grad}(f)$. Now, $\text{div}(F) = 0$ implies $\text{divgrad}(f) = \Delta f = 0$.

4 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x,y) = [3x^2 - 3y^2, -6xy] \; .$$

We have now all the derivatives together. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\text{grad}} 1$$

$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$

$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

If a surface S is parametrized as $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ over a domain R in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int\int_F \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dudv \; .$$

1 Compute the flux of $\vec{F}(x,y,z) = [0,1,z^2]$ through the upper half sphere S parametrized by

$$\vec{r}(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] \; .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u,v)) = [0,1,\cos^2(v)]$ so that

$$\int_0^{2\pi} \int_0^{\pi} -[0,1,\cos^2(v)] \cdot [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)] \; dudv \; .$$

The flux integral is $\int_0^{2\pi} \int_{\pi/2}^{\pi} -\sin^2(v)\sin(u) - \cos^3(v)\sin(v) \; dudv$ which is $-\int_{\pi/2}^{\pi} \cos^3 v \sin(v) \; dv = \cos^4(v)/4|_0^{\pi/2} = -1/4$.

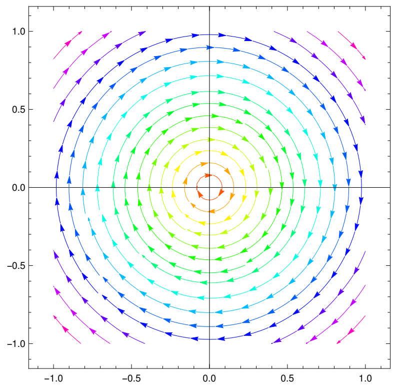
2 Calculate the flux of $\vec{F}(x,y,z) = [1,2,4z]$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r,\theta) = [r\cos(\theta), r\sin(\theta), r^2]$ where $\vec{r}_r \times \vec{r}_\theta = [-2r^2\cos(\theta), -2r^2\sin(\theta), r]$ and $\vec{F}(\vec{r}(u,v)) = [1,2,4r^2]$. We get $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3) \; drd\theta = 2\pi$.

3 Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot d\vec{S}$ for $\vec{F}(x,y,z) = [xy, yz, zx]$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,1] \times [0,1]$ and has an upward orientation. **Solution:** $\text{curl}(F) = [-y,-z,-x]$. The parametrization $\vec{r}(u,v) = [u, v, 4-u^2-v^2]$ gives $r_u \times r_v = [2u, 2v, 1]$ and $\text{curl}(F)(\vec{r}(u,v)) = [-v, u^2+v^2-4, -u]$. The flux integral is $\int_0^1 \int_0^1 [-2uv + 2v(u^2+v^2-4) - u] \; dvdu = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.

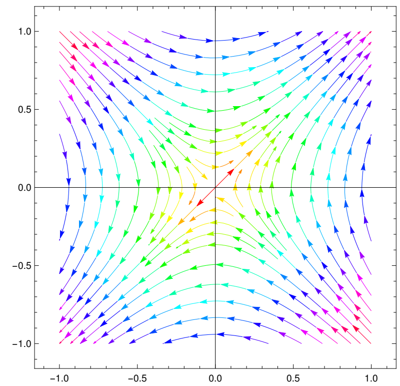
Lecture 29: Curl, Div, Flux

1 Which vector fields I-VI are incompressible?

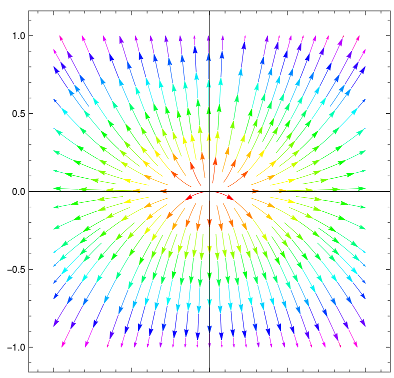
2 Which vector fields I-VI are irrotational?



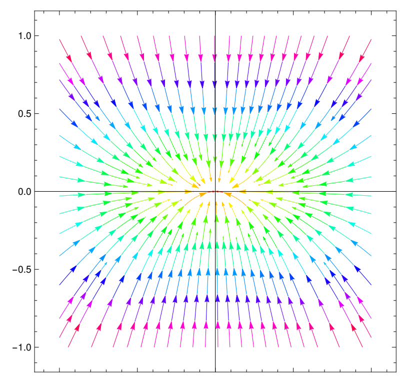
I



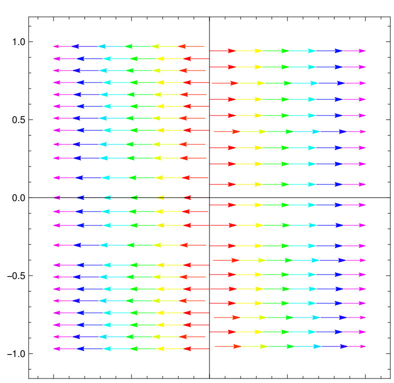
II



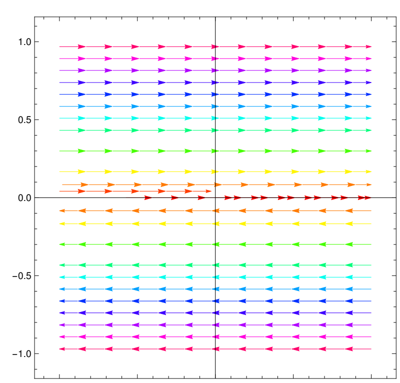
III



IV



V



VI

Lecture 29: Flux integrals

1 I have a detector surface $[u, v, 1 - u^2]$ with $0 \leq u \leq 1, 0 \leq v \leq 3$. A radiation field from a defective battery is of the form $\vec{F}(x, y, z) = [x^2, 3y, z]$. What is the flux of $\vec{F}$ through this surface?