Name:

MWF 9 Oliver Knill
MWF 9 Arnav Tripathy
MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
MWF 10:30 Karl Winsor
MWF 10:30 Drew Zemke
MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 and 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

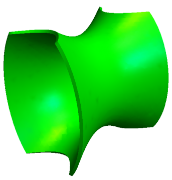
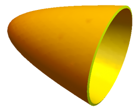
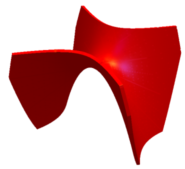
Problem 1) (20 points) No justifications are needed.

1)	T	F	The length of the vector $[3, 4, 0]$ is 25.
2)	T	F	The point $(x, y, z) = (1, 0, 0)$ has spherical coordinates $(\rho, \phi, \theta) = (1, \pi/2, 0)$.
3)	T	F	The distance of a point P to a line L contained in a plane Σ is larger or equal than the distance of the point P to the plane.
4)	T	F	The velocity vector of $\vec{r}(t) = [t, t, t]$ at time $t = 2$ is twice the velocity vector at time $t = 1$.
5)	T	F	If $\vec{u} \times \vec{v} = \vec{0}$, then $\mathrm{Proj}_{\vec{u}}(\vec{v}) \times \mathrm{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.
6)	T	F	If $\vec{u} \times \vec{v} = \vec{v} \times \vec{w}$, then $\vec{v} \cdot (\vec{u} \times \vec{w}) = 0$.
7)	T	F	There is a point not at the origin with Cartesian coordinates $(x, y, z) = (a, b, c)$ and spherical coordinates $(\rho, \theta, \phi) = (a, b, c)$.
8)	T	F	The two planes $2x + 2y - z = 4$ and $-4x - 4y + 2z = 3$ intersect in a line.
9)	T	F	If the distance between two points P and Q is zero, then $P = Q$.
10)	T	F	If the distance between two lines L and M is zero, then $L = M$.
11)	T	F	The surface $x^2 + y^2 + 4y = -z^2$ is a two-sheeted hyperboloid.
12)	T	F	There are two vectors $\vec{v}, \vec{w}$ in $\mathbf{R}^3$ of length 1 for which the dot product is 2.
13)	T	F	If the acceleration of a curve $\vec{r}(t)$ is zero at all times and the velocity is non-zero at time $t = 0$, then the curve is a line.
14)	T	F	The lines $\vec{r}(t) = [3t, 4t, 5t]$ and $\vec{s}(t) = [-4t, 3t, 0]$ intersect perpendicularly.
15)	T	F	The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axis.
16)	T	F	Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $\lvert\vec{u} \cdot \vec{v}\rvert\lvert\vec{w}\rvert = \lvert\vec{u}\rvert\lvert\vec{v} \cdot \vec{w}\rvert$.
17)	T	F	The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a cylinder.
18)	T	F	The arc length of the curve $[\sin(t), \cos(t), t]$ from $t = 0$ to $t = 2\pi$ is larger than 2π.
19)	T	F	The surface parametrized by $\vec{r}(u, v) = [v \sin(u), v \cos(u), 0]$ with $0 \leq u < 2\pi$, $v \geq 0$ is a plane.
20)	T	F	It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

	Total

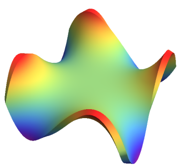
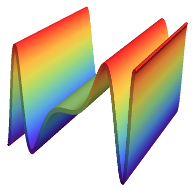
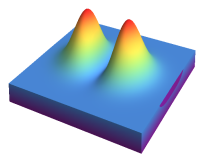
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the surfaces with their equations $g(x, y, z) = 1$. Enter O, if there is no match.

I  II  III 

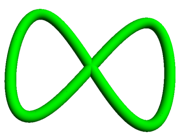
Function $g(x, y, z) =$	Enter O,I,II or III
$2x - y^2 - z^2$	
$2x^2 - y^2 + z^2$	
$2x - y$	
$2x^2 - y^2 - z$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

I  II  III 

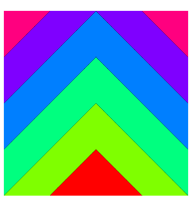
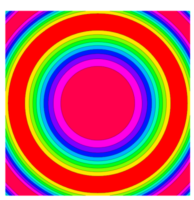
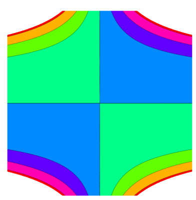
Function $f(x, y) =$	Enter O,I,II or III
$xy(x^2 - y^2)$	
$\sin(x^3)$	
$\sin(y^4)$	
$x^2 \exp(-x^2 - y^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

I  II  III 

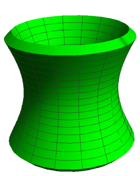
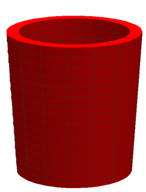
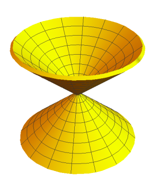
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$[t, \sin(4t), \cos(4t)]$	
$[\cos(t), \cos(t), \sin(2t)]$	
$[3t, 1 - t, 5t]$	
$[t\sin(t), t\cos(t), t]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Function $g(x, y) =$	Enter O, I,II or III
$\sin(x^2 + y^2)$	
$\sin(x) - y$	
$\|x\| + y$	
xy^2	

e) (2 points) Match the quadrics. Enter O if there is no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2 + y^2 - z^2 = 1$	
$x^2 + y^2 + z^2 = 1$	
$x^2 + y^2 = 1$	
$x^2 + y^2 = z^2$	

Problem 3) (10 points)	(Only answers are needed)

a) (3 points) Write the equations of a surface in Cartesian, Cylindrical and Spherical coordinates. The first row gives an example:

Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
$z = 1$	$z = 1$	$\rho\cos(\phi) = 1$
		$\rho\sin(\phi) = 1$
	$r\cos(\theta) = 1$	
$x^2 + y^2 + z^2 = 1$		

b) (3 points) Assume $\vec{u}, \vec{v}$ are unit vectors which are perpendicular. Check one box in each row:

The value	is larger than 0	is smaller than 0	is equal to 0
$\vec{u} \cdot \vec{v}$			
$\lvert\vec{u} \times \vec{v}\rvert$			
$\vec{u} \cdot (\vec{v} \times \vec{u})$			

c) (2 points) Complete the following table which uses the vectors $\vec{i} = [1, 0, 0], \vec{j} = [0, 1, 0], \vec{k} = [0, 0, 1], -\vec{i}, -\vec{j}, -\vec{k}$ or $\vec{0}$ in one of the first 3 boxes. Enter an scalar in each of the 3 boxes at the bottom.

$\vec{i} \times \vec{i} =$		$\vec{i} \times \vec{j} =$		$\vec{k} \times \vec{j} =$	
$\vec{j} \cdot \vec{i} =$		$\vec{j} \cdot \vec{j} =$		$\vec{j} \cdot \vec{k} =$	

d) Similarly as a pianist must practice etudes, or an athlete needs to push weights, a mathematician must practice basic computations. Our theme is build from the vectors

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} .$$

Please play as Allegro Sostenuto.

(1 point) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

(1 point) What is the projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

Problem 4) (10 points)

In a TED talk of 2013, **Raffaello D'Andrea** and a team demonstrated **"quadrotor athletes"**.

Assume the three robots are at positions

$$A = (2,3,1), B = (2,3,3) \text{ and } C = (5,4,2).$$

What is the area of the triangle they span?

A figure from an article by J. Sidman and A. St John in the Notices of the AMS, October 2017.

Problem 5) (10 points)

The six-legged **Gough-Stewart platform** has applications in flight simulators, robotics, crane technology, underwater research, telescopes and orthopedic surgery. The bottom positions of the legs are

$$A_1 = (5, -2, 0), A_2 = (5, 1, 0), A_3 = (-3, 5, 0),$$

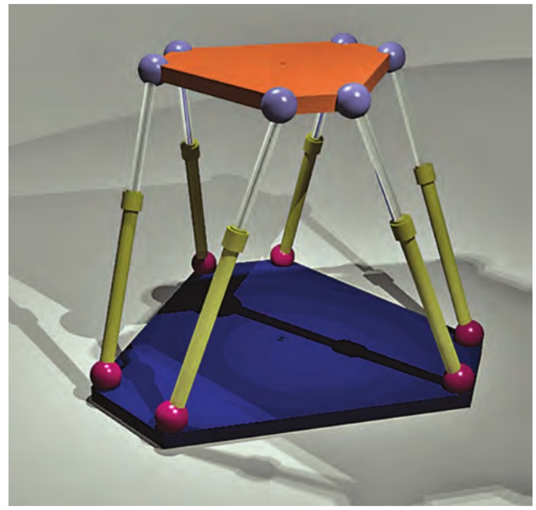
$$A_4 = (-5, 3, 0), A_5 = (-5, -3, 0), A_6 = (-3, -5, 0).$$

The top positions of the legs are

$$B_1 = (-5, 2, 6), B_2 = (-5, -1, 6), B_3 = (3, -5, 6),$$

$$B_4 = (5, -3, 6), B_5 = (5, 3, 6), B_6 = (3, 5, 6).$$

a) (5 points) What is the distance between B_1 and the plane containing $A_1, \dots, A_6$?
b) (5 points) What is the distance between B_1 and the line through A_1 and A_2?



Picture: Jessica Sidman and Audry St. John in the Notices of the AMS

Problem 6) (10 points)

Even though Saturn is much larger than the Earth, its gravitational force is only 7 percent larger than here on Earth. When **Cassini** plunged into Saturn, it felt an acceleration

$$\vec{r}\,''(t) = [\pi \sin(\pi t), 0, -10 - 2t] .$$

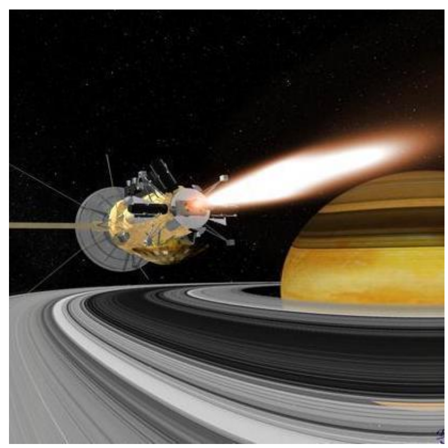
We know the initial velocity

$$\vec{r}\,'(0) = [2, 5, 1]$$

and initial position

$$\vec{r}(0) = [0, 0, 1000] .$$

Where is the spacecraft at $t = 1$?



Picture: NASA

Problem 7) (10 points)

Let's look at the two planes

$$x + y + z = 1$$

and

$$x + y - z = 1 .$$

a) (4 points) Find the plane $ax+by+cz = d$ through $P = (1,0,0)$ which is perpendicular to both.

b) (2 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1,0,0)$ which is contained in both planes.

c) (4 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1,0,0)$ which is contained in the first plane but not the second.

Problem 8) (10 points)

The world was supposed to end on September 23, 2017 due to the mysterious planetary system HD 7924. But here you sit and have to take the first hourly. A moon on HD 7924 moves on an epicycle

$$\vec{r}(t) = [10\cos(t), 10\sin(t), 0] + [2\cos(5t), 2\sin(5t), 0] \ .$$

a) (2 points) Find the velocity $\vec{r}\,'(0)$ at $t = 0$ and the speed $|\vec{r}\,'(0)|$ at $t = 0$.

b) (2 points) Find the acceleration $\vec{r}\,''(0)$ at $t = 0$.

c) (3 points) Find $\kappa(0) = |\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$.

d) (3 points) Inhabitants from HD 7924 beam you the hint $|\vec{r}\,'(t)|^2 = 400\cos^2(2t)$. Use this to find the arc length from $t = 0$ to $t = 2\pi$.

Will the world end on September 23?

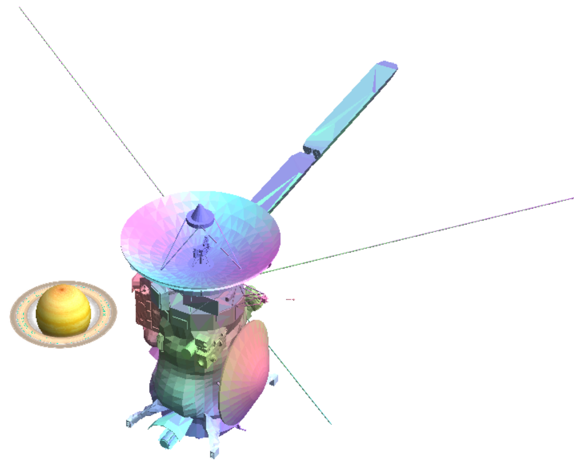
30 Updated on September 20, 2017 at 12:37 PM, Posted on September 20, 2017 at 11:03 AM

Viral videos and theories claim an apocalypse will begin on Sept. 23. This photo is an artist's impression of a view from the HD 7924 planetary system looking back toward our sun. (Karen Teramura; BJ Fulton, University of Hawaii, Institute for Astronomy.)

Fun fact: the guy who came up with the date September 23, 2017 has revised his estimate to October 15, 2017. At the time of taking the exam in 2017 there still had been hope for avoiding the second one.

Problem 9) (10 points)

Two weeks ago, in a grand finale, the Cassini space craft plunged into the atmosphere of **Saturn**. To build a model of the situation we have to parametrize various parts on the probe which were used both for measurement and communication.
You don't have to specify the parameter bounds but give the parametrizations for each of the 5 objects:



Picture: by Mathematica using a printable

3D STL models provided by NASA

a) (2 points) Saturn is a sphere $(x-1)^2 + y^2 + z^2 = 16$.

$\vec{r}(\theta, \phi) = [\quad , \quad , \quad]$.

b) (2 points) The rings are given by $z = 0, r^2 = x^2 + y^2 \leq 25$.

$\vec{r}(r, \theta) = [\quad , \quad , \quad]$.

c) (2 points) The satellite dish $(x-50)^2 + (y-70)^2 = z$ beams pictures back to earth.

$\vec{r}(x, y) = [\quad , \quad , \quad]$.

d) (2 points) There is also a satellite antenna of the form $(x-50)^2 + z^2 = 1/100$.

$\vec{r}(\theta, y) = [\quad , \quad , \quad]$.

e) (2 points) There is also a device of the form $(x-50)^2 + z^2 - (y-70)^2 = 1$.

$\vec{r}(\theta, y) = [\quad , \quad , \quad]$.