10/1/2019 FIRST HOURLY PRACTICE 2 **Math 21a, Fall 2019**

Name:

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MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
MWF 10:30 Karl Winsor
MWF 10:30 Drew Zemke
MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F If $|\vec{r}\,'(t)| = 1$ for all t, then the arc length of the curve defined by $\vec{r}(t)$ for $2 \leq t \leq 5$ is equal to 3.

Solution:
Use the arc length formula. We integrate 1 from $t = 2$ to $t = 5$. This is 3.

2) T F The length of the vector connecting $A = (3, 4, 5)$ to $B = (6, 8, 5)$ is 5.

Solution:
The length of the vector $\vec{AB}$ is $\sqrt{3^2 + 4^2} = 5$.

3) T F If $|\vec{v} + \vec{w}|^2 = 2\vec{v} \cdot \vec{w}$, then $\vec{v} = 2\vec{w}$.

Solution:
The statement actually implies that $\vec{v} = \vec{w} = [0, 0, 0]$.

4) T F The distance between two points P and Q is smaller than or equal to $|\vec{OP}|+$ $|\vec{OQ}|$, where $O = (0, 0, 0)$ is the origin.

Solution:
This is the triangle inequality. If you make a detour over a third point, the distance gets bigger.

5) T F The line $\vec{r}(t) = [5 + 2t, 2 + t, 3 + t]$ is located on the plane $x - y - z = 0$.

Solution:
Just plug in $x = 5 + 2t, y = 2 + t, z = 3 + t$ into the equation.

6) T F The arc length of a circle of radius r is πr^2.

Solution:
We either know that or compute it.

7) T **F** The surface $x^2 - y^2 + 4y = z^2 + 2z$ is an elliptic paraboloid.

Solution:
Complete the square. It is a hyperbolic paraboloid.

8) T **F** For any two vectors, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| + |\vec{w}|$.

Solution:
It is not plus on the right hand side, but times.

9) **T** F The acceleration of $\vec{r}(t) = [t, t, t]$ is $[0, 0, 0]$ everywhere.

Solution:
Indeed, the second derivative is $[0, 0, 0]$.

10) T **F** If $\vec{v} = \vec{PQ} = [2, 1, 1]$ then $|\vec{v}|$ is larger than the distance between P and Q.

Solution:
It is equal.

11) T **F** If $\vec{v} \cdot \vec{w} > 0$, then the angle between $\vec{v}$ and $\vec{w}$ is larger than $90°$.

Solution:
Use the cos formula. Positive means actually that the angle is acute.

12) **T** F If $[5, 6, 4] \times \vec{x} = \vec{x}$ then $\vec{x} = [0, 0, 0]$.

Solution:
Yes, since $\vec{x}$ must be perpendicular to $\vec{x}$.

13) | T | F | The curve $\vec{r}(t) = [t-1, 1-t, t]$ has constant acceleration $[0, 0, 0]$.

Solution:
Just differentiate

14) | T | F | The line $\vec{r}(t) = t[7, 7, 1]$ hits the plane $-x - y = 7z$ in a right angle.

Solution:
The normal vector is $[-1, -1, 7]$.

15) | T | F | The surface given in spherical coordinates as $\sin(\phi) = 1/\rho$ is a cylinder.

Solution:
It is on the equator

16) | T | F | Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \cdot \vec{v})(\vec{w} \cdot \vec{u})| \leq |\vec{u}|^2 |\vec{v}||\vec{w}|$.

Solution:
Use the cos identity for the length of the dot product

17) | T | F | The surface given in spherical coordinates as $\rho^2 = 1$ is a sphere.

Solution:
Look at the traces

18) | T | F | The arc length of the curve $[5\sin(t), 1, 5\cos(t)]$ from $t = 0$ to $t = 2\pi$ is equal to 10π.

Solution:
It is a circle

19) | T | F | The surface parametrized by $\vec{r}(u,v) = [u^5 - v^5, u^5 + v^5, u^5]$ is a plane.

Solution:
$x + y = 2z$

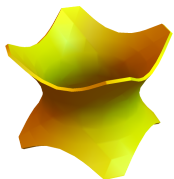
20) | T | F | It can happen that the cross product of two non-zero vectors $\vec{v}$ with $\vec{w}$ is parallel to $\vec{v}$.

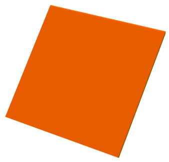
Solution:
It can be that $\vec{v} = [1,0,0] = \vec{w}$. In that case the cross product is the zero vector which is parallel to $\vec{v}$.

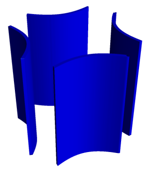
| | Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.

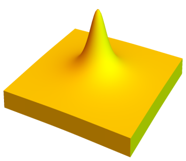

I

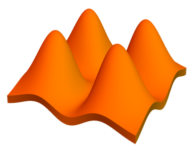

II

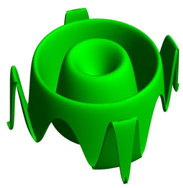

III

Function $g(x, y, z) = 1$	Enter O,I,II or III
$x^2 + y^2 - z^2$	
x^2y^2	
$x - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.


I

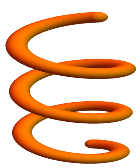

II


III

Function $f(x, y) =$	Enter O,I,II or III
$1/(1 + x^2 + y^2)$	
$x^2y^2e^{-x^2-y^2}$	
$\sin(x^2 + y^2)$	
$\cos(y)$	

c) (2 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.


I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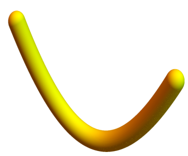
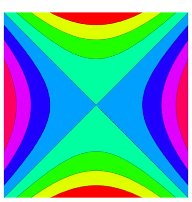
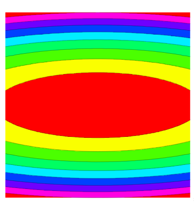

III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$[t, t\sin(3t), t\cos(3t)]$	
$[\cos(t), \sin(t), \cos(t)]$	
$[\sin(t), \cos(t), t]$	
$[t, t, t^2]$	

d) (2 points) Match functions g with contour plots in the xy-plane. Enter O, if there is no match.

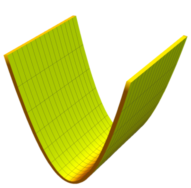

I

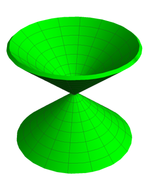

II


III

Equation	Enter O, I,II or III
$\sin(x) - y$	
$x^2 - y^2$	
$10x^2 + y^2$	
$x^2 + 10y^2$	

e) (2 points) Match the quadrics. Enter O if no match.


I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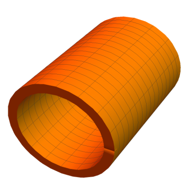

III

Quadric	Enter O,I,II or III
$x^2 = z$	
$x^2 + y^2 = 1$	
$xy = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

Solution:
a) I,III,II,O
b) I,II,III,O
c) I,O,II,III
d) III,I,O,II
e) I,III,O,O,II

Problem 3) (10 points)	(Only answers are needed)

a) (4 points) Mark what applies for any two vectors $\vec{v}$ and $\vec{w}$ in space.

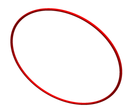
Object	always 0	can be $\neq 0$	always $\vec{0} = [0,0,0]$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				
$(\vec{v} \times \vec{w}) \cdot \vec{v}$				
$(\text{proj}_{\vec{v}}\vec{w}) \times \vec{w}$				
$(\text{proj}_{\vec{w}}\vec{v}) \cdot \vec{w}$				

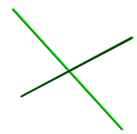
b) (3 points) **Conic sections** (parabolas, ellipses or hyperbolas) can be seen as intersections of a two dimensional cone

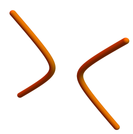
$$x^2 + y^2 = z^2$$

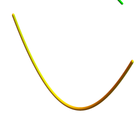
with a 2D plane. Identify the quadrics in the following three cases:

Intersect with plane	Enter A-D
$z = 1$	
$z = \sqrt{2}x$	
$x = y + 1$	

A 

B 

C 

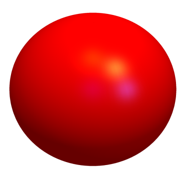
D 

c) (3 points) Three dimensional cone is given by the equation

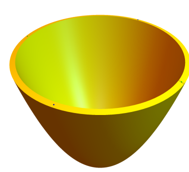
$$x^2 + y^2 + z^2 = w^2$$

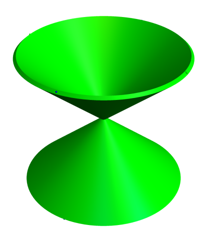
in four dimensional space. If we intersect it with a three dimensional space, we get quadrics. We want you to identify a few quadrics. In the pictures the quadrics might be turned or scaled. You get a point for every right answer, meaning that you can miss one and still have full credit.

Intersect with the 3D plane	Enter A-D or O
$w = 0$	
$w = z + 1$	
$y = z$	
$w = 1$	

A 

B 

C 

D 

Solution:
a)

Object	always 0	can be $\neq 0$	always $\vec{0} = [0,0,0]$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				*
$(\vec{v} \times \vec{w}) \cdot \vec{v}$	*			
$(\text{proj}_{\vec{v}}\vec{w}) \times \vec{w}$				*
$(\text{proj}_{\vec{w}}\vec{v}) \cdot \vec{w}$		*		

b) A,B,C
c) O,C,D,A.

Problem 4) (10 points)

We are going into the **furniture design business**. Our first task is to construct a chair. Find the distance between the lines spanned by the parallel arm rests AB and CD, where

$$A = (1,0,2), B = (4,1,2)$$

and

$$C = (3,3,3), D = (6,4,3) \ .$$

Solution:
Since the two lines are parallel, we can not just use blindly the distance formula for two lines. But we can take one point on the first line and compute the distance to the other. The distance is therefore given by

$$|\vec{AB} \times \vec{AC}|/|\vec{AB}| .$$

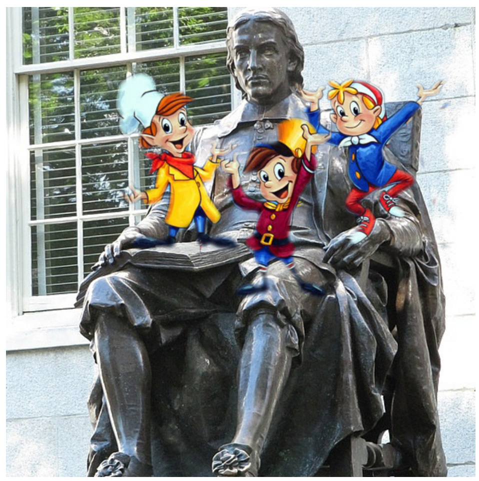
As $\vec{AB} = [3,1,0]$ and $\vec{AC} = [2,3,1]$, we have $\vec{AB} \times \vec{AC} = [1,-3,7]$. So $|\vec{AB} \times \vec{AC}|/|\vec{AB}| = |[1,-3,7]|/|[3,1,0]| = \boxed{\sqrt{59}/\sqrt{10}}$.

Problem 5) (10 points)

a) (6 points) The first derivative of parametrized curve is called **"velocity"**. You have also learned the terms **"acceleration"** and maybe **"jerk"** for the second and third derivative. Less well known are **"snap"**, **"crackle"**, **"pop"** for the fourth, fifth and sixth derivatives. Since we could not yet find the seventh derivative named, let's call it the **"Harvard"**. Compute the **"Harvard"** of the curve

$$\vec{r}(t) = [\cos(2t) + t, \sin(2t), t^2]$$

at time $t = 0$.



b) (4 points) The parametrization $\vec{v}(t) = \vec{r}\,'(t)$ defines a new curve. It is located on a surface. Which of the following surface is it?

	Check one
cylinder	A
cone	B
plane	C
ellipsoid	D
paraboloid	E

Solution:
a) Start differentiating:
$\vec{r}\,'(t) = [-2\sin(2t) + 1, 2\cos(2t), 2t)]$. velocity
$\vec{r}\,''(t) = [-4\cos(2t), -4\sin(2t), 2)]$. acceleration
$\vec{r}\,'''(t) = [8\sin(2t), -8\cos(2t)0)]$. jerk
$\vec{r}\,''''(t) = [16\cos(2t), 16\sin(2t)0)]$. snap
$\vec{r}\,'''''(t) = [-32\sin(2t), 32\cos(2t)0)]$. crackle
$\vec{r}\,''''''(t) = [-64\cos(2t), -32\sin(2t)0)]$. pop
$\vec{r}\,'''''''(t) = [128\sin(2t), -64\cos(2t)0)]$. Harvard
$\vec{r}\,'''''''(0) = [0, -128, 0]$.
b) $\vec{r}\,'(t) = [-2\sin(2t) + 1, 2\cos(2t), 2t)]$.
The surface is located on a cylinder as $(x-1)^2 + y^2 = 4$. The answer is $\boxed{\text{A}}$.

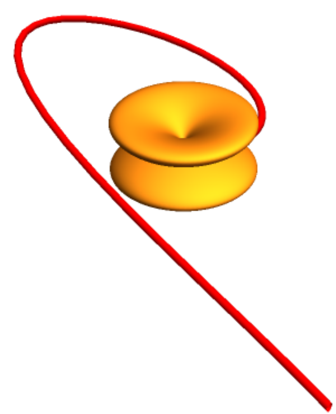
Problem 6) (10 points)

A kid plays with a **Yo-Yo**. It is accelerated periodically with $\vec{r}\,''(t) = [\sin(t), 0, \cos(t) - 10]$. Find the position of the Yo-Yo at time $t = 2\pi$ if the initial position is

$$\vec{r}(0) = [5, 5, 0]$$

and the initial velocity is

$$\vec{r}\,'(0) = [1, 1, 1]\ .$$



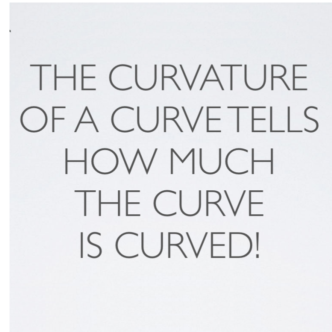
Solution:
a) Just integrate and fix the constant $\vec{r}\,'(t) = [-\cos(t), 0, \sin(t) - 10t] + [2, 1, 1]$.
Now integrate again: $\vec{r}(t) = [-sin(t) + 2t + 5, t + 5, \cos(t) - 5t^2 + t + 1]$. Evaluated at $t = 2\pi$ this is $\vec{r}(2\pi) = [4\pi + 5, 2\pi 5, -20\pi^2 + 2\pi]$.

Problem 7) (10 points)

a) (5 points) Compute the arc length of the curve

$$\vec{r}(t) = [t, t^2, 2t^3/3]$$

if $0 \leq t \leq 4$.



b) (5 points) What is the following expression

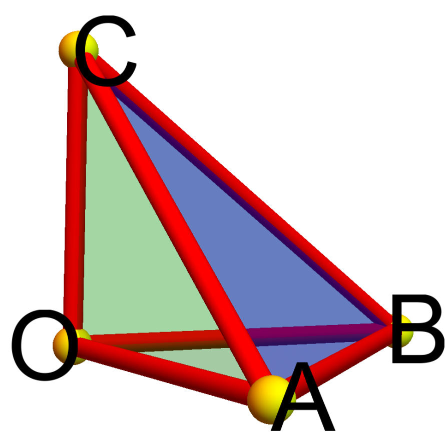
$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

called "curvature" of this curve at $t = 0$?

Solution:
a) $\vec{r}\,'(t) = [1, 2t, 2t^2]$. $|\vec{r}\,'(t)| = 2t^2 + 1$. Integration $\int_0^4 2t^2 + 1\, dt$ gives $140/3$.
b) The curvature is $|[1, 0, 0] \times [0, 2, 0]|/|[1, 0, 0]|^3$. This is $\boxed{2}$.

Problem 8) (10 points)

The triangle ABC is obtained by slicing a corner ABC off from a cube. One obtains a so called **trirectangular tetrahedron**.



a) (5 points) Find the square of the area of the triangle ABC, where $A = (3, 0, 0), B = (0, 6, 0), C = (0, 0, 8)$.

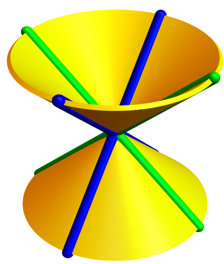
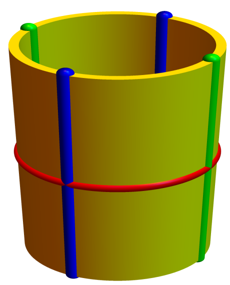
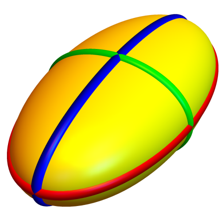
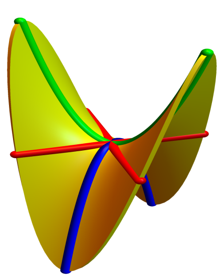
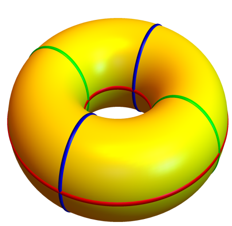
b) (5 points) Compute also the sum of the squares of the areas of the triangles OAB, OBC and OCA, where $O = (0, 0, 0)$ is the origin. The sum should get the same value you got in a).

P.S. the same computation can be repeated for arbitrary points $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$. It proves a not so well known theorem telling that the sum of the squares of the side wall areas is the square of the face area. It is a 3 dimensional version of Pythagoras and also goes by the name **de Gua theorem** or **Faulhaber theorem**.

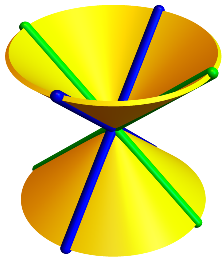
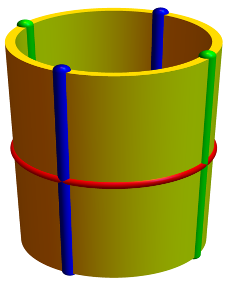
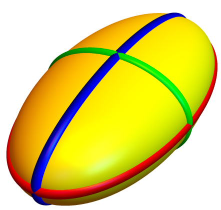
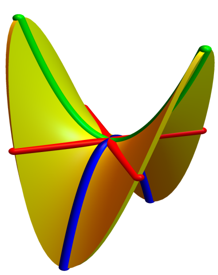
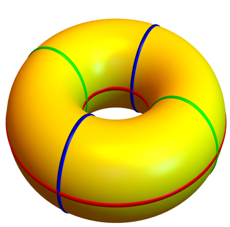
Solution:
a) The area is half the area of a parallelogram. Its square is therefore $|AB \times AC|^2/4$ which is $|[3,-6,0] \times [3,0,-8]|^2/5$. This is $\boxed{801}$.
b) The area of a side triangle is half the area of a square. The sum of the areas squared is $18^2/4 + 48^2/4 + 24^2/4 = 801$.
P.S. The full computation is the same $[a,-b,0] \times [a,0,-c] = [bc, ac, ab]$ which has square length $(bc)^2 + (ac)^2 + (ab)^2$. The triangle has 1/4'th of this. The area of the triangle is therefore $(bc)^2/4 + (ac)^2/4 + (ab)^2/4$ and each summand is the square area of a side triangle.

Problem 9) (10 points)

The 3D printing venture "Math-Candy" (math-candy.com) asks you to do some product development. In each of the 5 following parametrizations, two entries are still missing, each entry being worth one candy (1 point).

a)		The surface $\boxed{x^2+y^2=z^2}$ is parametrized by $\vec{r}(\theta,z)=[\ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z]$
b)		The surface $\boxed{x^2+y^2=1}$ is parametrized by $\vec{r}(\theta,z)=[\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, z]$
c)		The surface $\boxed{2(x-1)^2+(y-5)^2+4z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=[\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, \cos(\phi)/2]$
d)		The surface $\boxed{x^2-y^2=z}$ is parametrized by $\vec{r}(x,y)=[\ldots\ldots\ldots, \ldots\ldots\ldots, x^2-y^2]$
e)		The surface $\boxed{(\sqrt{x^2+y^2}-2)^2+z^2=1}$ is parametrized by $\vec{r}(\theta,\phi)=[(2+\cos(\phi))\cos(\theta), \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots]$

Solution:

a)		The surface $\boxed{x^2 + y^2 = z^2}$ is parametrized by $\vec{r}(\theta, z) = [z\cos(\theta), z\sin(\theta), z]$
b)		The surface $\boxed{x^2 + y^2 = 1}$ is parametrized by $\vec{r}(\theta, z) = [\cos(\theta), \sin(\theta), z]$
c)		The surface $\boxed{2(x-1)^2 + (y-5)^2 + 4z^2 = 1}$ is parametrized by $\vec{r}(\theta, \phi) =$ $[1 + \sin(\theta)\cos(\theta)/\sqrt{2}, 5 + \sin(\theta)\sin(\phi), \cos(\phi)/2]$
d)		The surface $\boxed{x^2 - y^2 = z}$ is parametrized by $\vec{r}(x, y) = [x, y, x^2 - y^2]$
e)		The surface $\boxed{(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1}$ is parametrized by $\vec{r}(\theta, \phi) = [(2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi)]$