11/5/2019 Second Hourly **Math 21a, Fall 2019**

Name:

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MWF 9 Tina Torkaman
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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No functions were harmed in the making of this exam.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) T F The chain rule assures that $\frac{d}{dt} f(\vec{r}(t))$ is the dot product of the gradient of f at $\vec{r}(t)$ and the velocity of $\vec{r}(t)$.

2) T F If $f(x, y) = c$ is a curve defining $y = y(x)$ near $(x, y) = (1, 1)$ then $y'(1) = -f_x(1, 1)/f_y(1, 1)$.

3) T F For $\vec{v} = \nabla f/|\nabla f(1, 1, 1)|$, the directional derivative $D_{\vec{v}} f(1, 1, 1)$ is a vector normal to the level surface of f passing through $(1, 1, 1)$.

4) T F There exists a function $f(x, y)$ without any critical points.

5) T F The function $f(x, y) = 6x + 17x^2 - 3y^2$ has a global maximum under the constraint $x = 0$.

6) T F If $f(x, y)$ is a function satisfying $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) > 0$, then f has a minimum at $(0, 0)$.

7) T F The point $(1, 0)$ is the only critical point of $f(x, y) = xy - y$.

8) T F If $\vec{r}(u, v)$ parametrizes a level surface S then $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface $\vec{r}(u, v)$ for all (u, v).

9) T F The surface area of a surface parametrized by $\vec{r}(u, v)$ over a domain R in the uv-plane is given by the integral $\int\int_R |\vec{r}_u \cdot \vec{r}_v| \, dudv$.

10) T F If $f_{xy}(x, y) = 0$ for all (x, y) in the plane, then all critical points of f are saddle points.

11) T F If $(0, 0)$ is a saddle point for $f(x, y)$ with discriminant $D < 0$, then the directional derivative $D_{\vec{v}} f(0, 0)$ can take positive and negative values as $\vec{v}$ ranges over all unit vectors.

12) T F If $(0, 0)$ is a saddle point for f, then $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ can have the same signs.

13) T F For $\vec{u} = [1, 0]$ and $\vec{v} = [0, 1]$ the discriminant D appearing in the second derivative test satisfies $D = (D_{\vec{u}}(D_{\vec{u}} f))(D_{\vec{v}}(D_{\vec{v}} f)) - (D_{\vec{u}} D_{\vec{v}} f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.

14) T F If $\vec{r}(t)$ is a curve on the surface $f(x, y, z) = 1$, then the velocity vector $\vec{r}\,'(t)$ is perpendicular to $\nabla f(\vec{r}(t))$ for all t.

15) T F The function $f(x, y) = 4x^3y^2 + \sin(\sin(x^{10})) + \cos(y^8)$ solves the partial differential equation $f_{xyxyxy} = 0$.

16) T F If $f_x(x, y) = f_y(x, y)$ for all x, y, then $f_{xx}(x, y) = f_{yy}(x, y)$ for all (x, y).

17) T F The linearization of $f(x, y) = x^2 + y^2$ at $(1, 2)$ is $L(x, y) = 2x(x-1) + 2y(y-2)$.

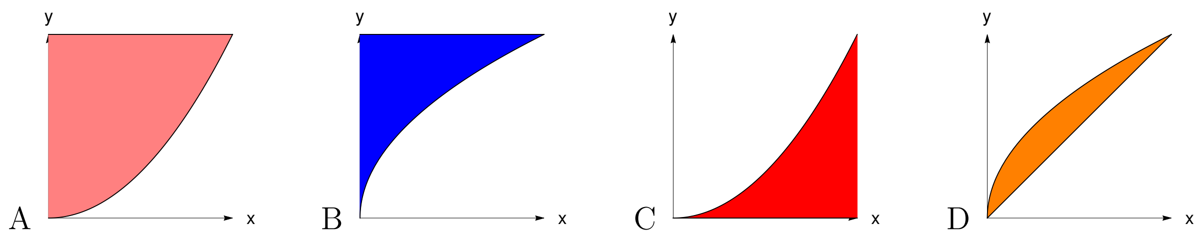
18) T F If $f(x, t)$ solves the partial differential equation $f_t = f_{xx}$, then $g(x, t) = f(x, -t)$ solves the partial differential equation $g_{tt} = g_{xx}$.

19) T F There is a function f such that $D_{\vec{u}} f(0, 0)$ is positive for all unit vectors $\vec{u}$.

20) T F The surface area of $z = x^2 + y^2$ with $x^2 + y^2 \leq 1$ does not depend on the parametrization.

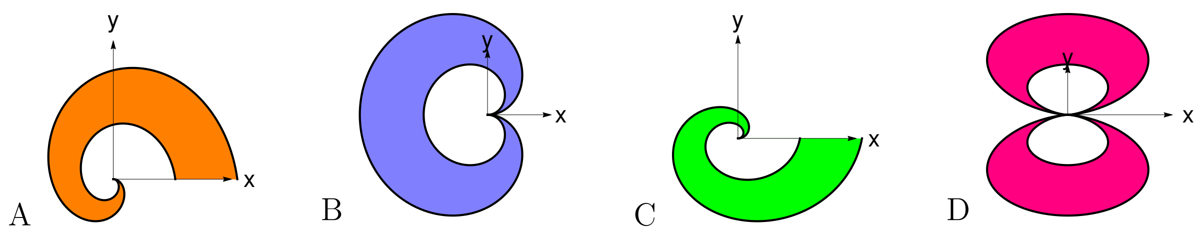
Problem 2) (10 points) No justifications are needed

a) (4 points) Match the following regions with their area computation. Each of the A-D will be used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_x^{\sqrt{x}} 1\ dydx$
	$\int_0^1 \int_{\sqrt{x}}^1 1\ dydx$
	$\int_0^1 \int_{\sqrt{y}}^1 1\ dxdy$
	$\int_0^1 \int_0^{\sqrt{y}} 1\ dxdy$

b) (4 points) Match the following regions with their area computation. Each of the A-D will be used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{\theta}^{2\theta} r\ drd\theta$
	$\int_0^{2\pi} \int_{\pi-\theta/2}^{2\pi-\theta} r\ drd\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r\ drd\theta$
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r\ drd\theta$

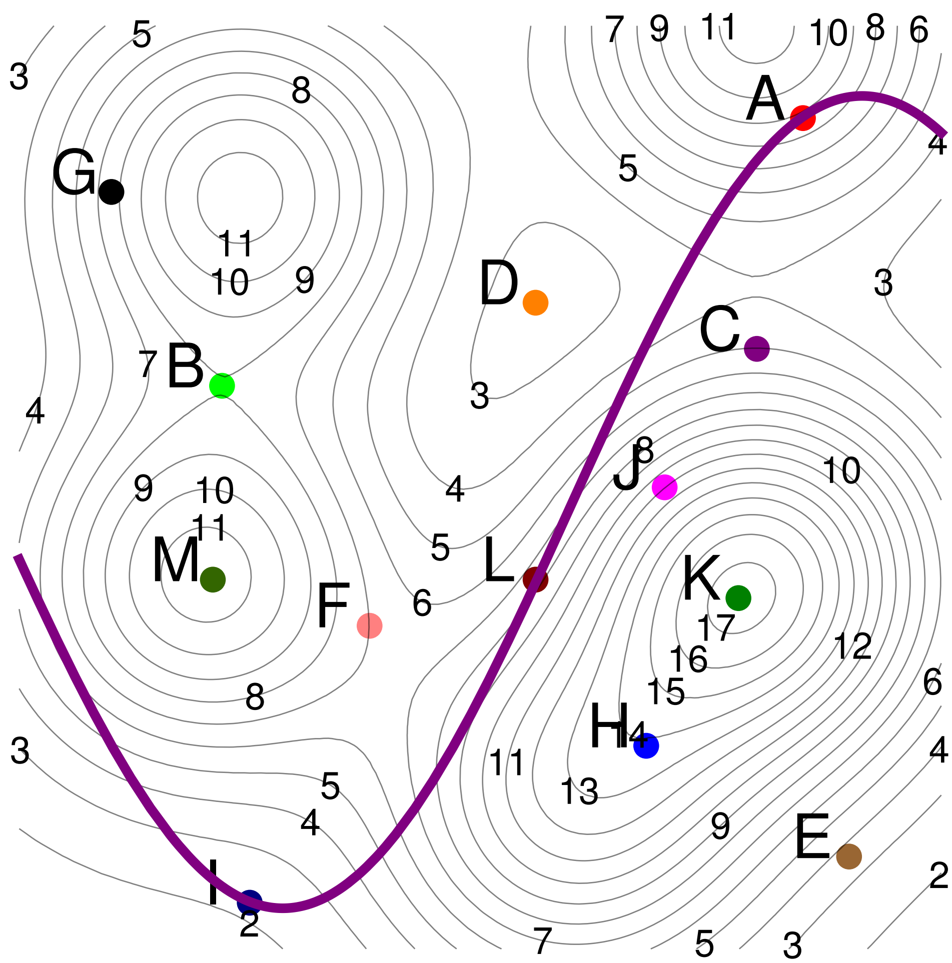
c) (2 points) Name all the four PDE's for a function $A(B,C)$ of variables B and C. Some equations come here with a constant parameter ν.

$A_{BB} + A_{CC} = 0$	$A_B + AA_C = \nu A_{CC}$	$A_{BB} = A_{CC}$	$A_B = \nu A_{CC}$

Problem 3) (10 points) (No justifications are needed.)

(10 points) In the diagram below, the thin curves are the level curves = contours of an unknown smooth function, $f(x, y)$. The thick line is the constraint curve, $g(x, y) = y - \sin(x) = 0$. For each of the descriptions below, write clearly in the blank the label of the point in the diagram that best matches the description. You should use each of the labels A-M only once and leave out the three labels which do not match.

	Enter A-M
A local but not global maximum of $f(x, y)$.	
A global maximum of $f(x, y)$.	
The point among A-M with maximal $\|\nabla f\|$.	
A saddle point of f.	
A local minimum $f(x, y)$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	



Problem 4) (10 points)

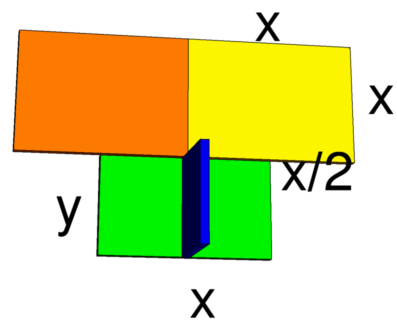
a) (5 points) The area function

$$f(x, y) = 2x^2 + xy + y^2$$

of an airplane has exactly one minimum under under the constraint

$$g(x, y) = 6x + 5y = 32 \ .$$

(You don't need to check this.) Find the minimum (x, y) using the method of Lagrange multipliers.



b) (5 points) Use the method of Lagrange multipliers to find (x, y) which maximizes the contour length $g(x, y) = 6x + 5y$ under the constraint $f(x, y) = 32$.

Problem 5) (10 points)

Define the **duck function**

$$f(x, y) = 4x^2 - y^2 - 2x^4 + 2y \ .$$

a) (8 points) Find local maxima, local minima and saddle points of f using the second derivative test.

b) (1 point) Does $f(x, y)$ have a global maximum on the entire plane? No justification is needed.

c) (1 point) Does $f(x, y)$ have a global minimum on the entire plane? No justification is needed.

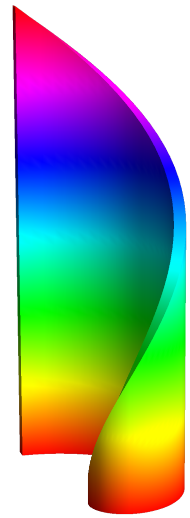
Any resemblance in this figure to actual persons or functions, living or dead is entirely coincidental.

Problem 6) (10 points)

a) (7 points) Set up the integral for the surface area of

$$\vec{r}(u,v) = \left[\ u\cos(u), u\sin(u), uv\ \right]$$

with $0 \leq v \leq 3$ and $0 \leq u \leq 4$. Simplify your answer as much as you can.



b) (3 points) Evaluate the integral.

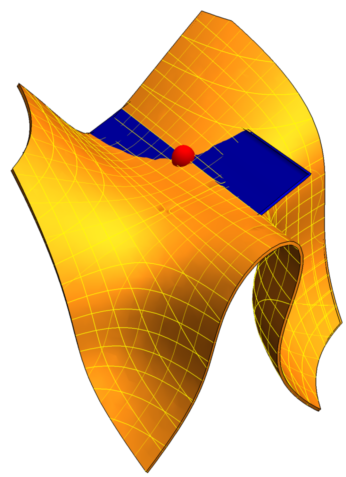
Problem 7) (10 points)

Knowing tangent planes can be important in **ray tracing**, where the computer bounces light around in the scene and reflects light at surfaces.

a) (6 points) Find the tangent plane to the surface

$$7xy^2z - x^2y + 2z^3 = 8$$

at the point $(1,1,1)$.



b) (4 points) Near $(x,y,z) = (1,1,1)$, the surface can be written as $z = z(x,y)$. Find the gradient $[z_x(1,1), z_y(1,1)]$ at $(1,1)$.

Problem 8) (10 points)

We have a mystery function f which satisfies $f(0,0) = 21$. We also know the two directional derivatives

$$D_{\vec{v}} f(0,0) = \frac{10}{5}, \quad D_{\vec{w}} f(0,0) = \frac{11}{5} ,$$

for the unit vectors $\vec{v} = [3,4]/5$ and $\vec{w} = [4,3]/5$.

a) (5 points) Find the tangent line to the curve $\{f(x,y) = 21\}$ at $(0,0)$.

b) (5 points) Estimate $f(-0.01, 0.03)$.

Problem 9) (10 points)

Compute the following double integrals:

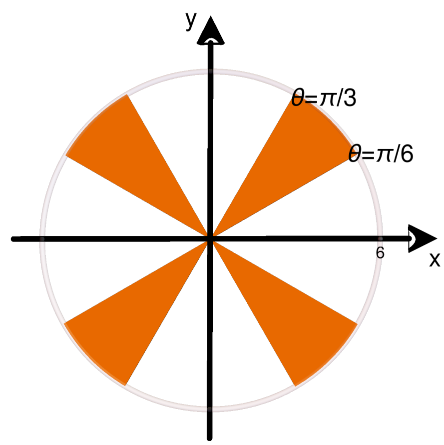
a) (5 points)

$$\iint_G x^2 + y^2 \; dxdy ,$$

where G is the propeller region given by

$$\{x^2 + y^2 \leq 36, y^2/3 < x^2 < 3y^2\} .$$

The propeller region consists of four blades, as you see in the picture.



b) (5 points)

$$\int_1^2 \int_0^{\sqrt{2-y}} \frac{\sin(\pi x)}{1 - x^2} \; dxdy .$$