**11/5/2019 Second Hourly Practice 4** **Math 21a, Fall 2019**

| Name: |
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| MWF 9 Oliver Knill |
| MWF 9 Arnav Tripathy |
| MWF 9 Tina Torkaman |
| MWF 10:30 Jameel Al-Aidroos |
| MWF 10:30 Karl Winsor |
| MWF 10:30 Drew Zemke |
| MWF 12 Stepan Paul |
| MWF 12 Hunter Spink |
| MWF 12 Nathan Yang |
| MWF 1:30 Fabian Gundlach |
| MWF 1:30 Flor Orosz-Hunziker |
| MWF 3 Waqar Ali-Shah |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
| --- | --- | --- |
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) True/False questions (20 points), no justifications needed

1) T F The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $|D_{\vec{v}} f(0,0)|$ among all unit vectors $\vec{v}$.

2) T F The relation $f_{xxyyxx} = f_{xxyxyx}$ holds everywhere for $f(x,y) = \cos(\exp(x^{10}) + \sin(x-y))$.

3) T F $\int_0^4 \int_0^{4x} f(x,y)\, dydx = \int_0^{16} \int_{y/4}^{16} f(x,y)\, dxdy$.

4) T F $g(x,y) = \int_y^0 \int_0^x f(s,t)\, dsdt$ satisfies $g_{xy} = -f(x,y)$.

5) T F If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$.

6) T F If $D_{[1/\sqrt{2},1/\sqrt{2}]} f(a,b) = 3$ and $D_{[1/\sqrt{2},-1/\sqrt{2}]} f(a,b) = 5$, then $D_{\vec{v}} f(a,b) \geq 0$ for all unit directions $\vec{v}$.

7) T F Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x,y)$ we have $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.

8) T F If $u(t,x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant $c$.

9) T F If the Lagrange multiplier $\lambda$ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.

10) T F The equation $f_x^2 + f_y^2 + f_z^2 = 1$ is an example of a partial differential equation.

11) T F If the discriminant $D$ of $f(x,y)$ is zero at $(0,0)$ then $\nabla f(0,0) = [0,0]$.

12) T F If $f(x,y,z) = 0$ describes the unit sphere, then the gradient $\nabla f$ points outwards.

13) T F If $f(x,y)$ is a continuous function then $\int_0^2 \int_0^1 f(x,y) \quad dxdy = \int_0^2 \int_0^1 f(y,x)\, dxdy$.

14) T F The point $(5,5,5)$ is a critical point of $f(x,y,z) = x + y + z$.

15) T F Assume $\nabla f(0,0) = [0,0]$ with discriminant $D > 0$, then $-f(x,y)$ has the same critical point $(0,0)$ with discriminant $D < 0$.

16) T F $\int\int_R |\nabla f|^2\, dxdy$ is the surface area of the cubic paraboloid $z = f(x,y) = x^3 + y^3$ defined over the region $R$.

17) T F If $D(x,y)$ is the discriminant of $f$ at $(x,y)$ then the following poetic formula of the **d**irectional **d**erivative of the **d**iscriminant holds: $D_{[1,0]} D = \partial_x D$.

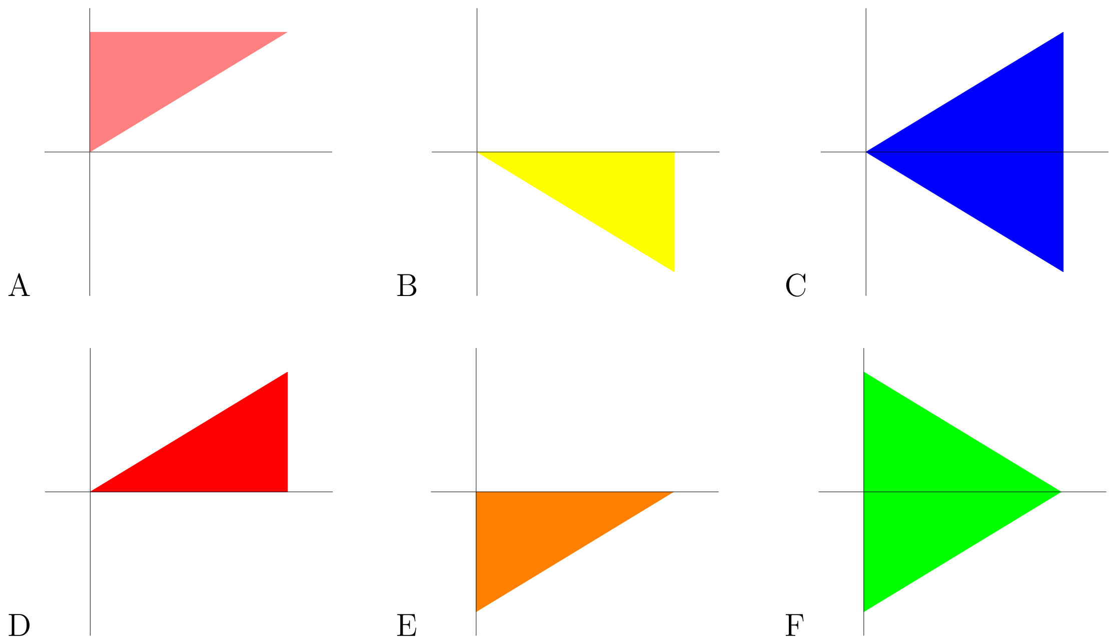
18) T F Assume $f(x,y) = -x^2 + y^4$ and a curve $\vec{r}(t)$ satisfying $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$ for all $t$.

19) T F The Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y) = c$ have the same solutions as the Lagrange equations for extremizing $F = f + g$ under the constraint $g = c$.

20) T F If $f$ is a maximum under the constraint $g = 1$ at $(0,0)$, and $(0,0)$ is not a critical point for both $f$ and $g$, then the level curves of $f$ and $g$ have the same tangent line at $(0,0)$.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-1}^{1} \int_{\vert y \vert}^{1} f(x,y)\ dxdy$ |
| | $\int_{0}^{1} \int_{0}^{x} f(x,y)\ dydx$ |
| | $\int_{-1}^{0} \int_{-y}^{1} f(x,y)\ dxdy$ |
| | $\int_{0}^{1} \int_{x-1}^{0} f(x,y)\ dydx$ |
| | $\int_{0}^{1} \int_{x-1}^{1-x} f(x,y)\ dydx$ |
| | $\int_{0}^{1} \int_{x}^{1} f(x,y)\ dydx$ |

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

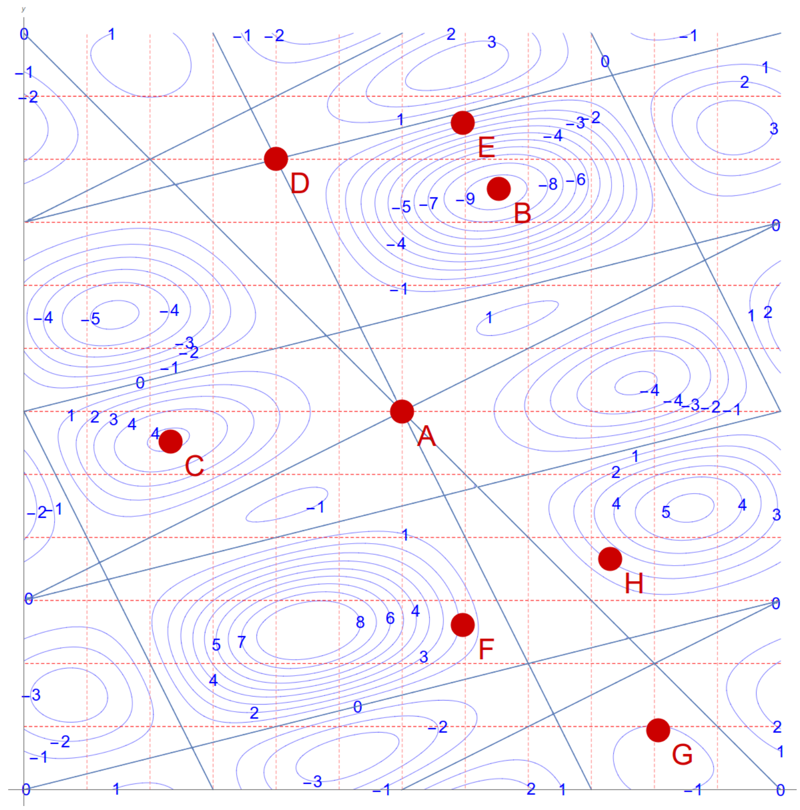
| Fill in 1-4 | Name |
|---|---|
| | Laplace |
| | Wave |
| | Transport |
| | Heat |

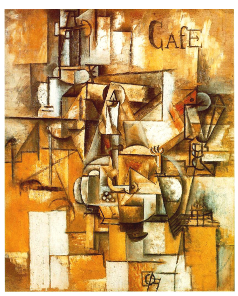
| Equation number | Formula for PDE |
|---|---|
| 1 | $\frac{\partial}{\partial_t} u - \frac{\partial}{\partial_y} u = 0$ |
| 2 | $\frac{\partial}{\partial_t} u - \frac{\partial^2}{\partial_x^2} u = 0$ |
| 3 | $\frac{\partial^2}{\partial_t^2} u - \frac{\partial^2}{\partial_x^2} u = 0$ |
| 4 | $\frac{\partial^2}{\partial_x^2} u + \frac{\partial^2}{\partial_y^2} u = 0$ |

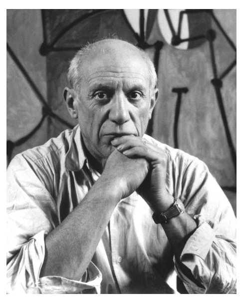
Problem 3) (10 points)

a) (7 points) The following contour map is inspired by a cubistic style of Picasso. Each of the points A-H fit exactly once

| | |
|---|---|
| A point where $f_x \neq 0, f_y = 0$ | |
| A saddle point of $f$ | |
| A local maximum of $f$ | |
| A critical point with $D = 0$ | |
| A local minimum | |
| A point with $f_y \neq 0, f_x = 0$ | |
| $\lvert\nabla f\rvert$ maximal among A-H | |
| Point where $D_{[-1,1]/\sqrt{2}} f = 0$ | |



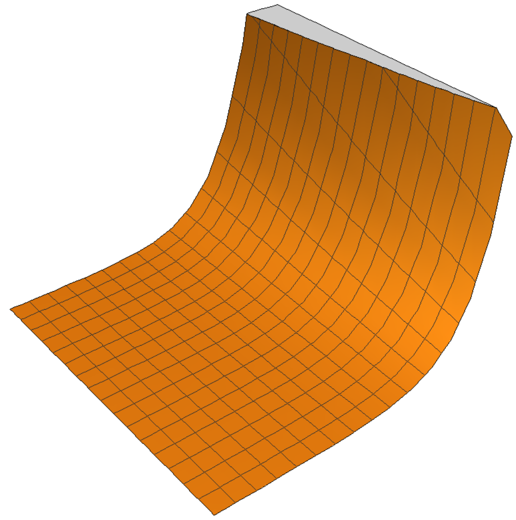


*The painting "Pigeon with Green Peas" by Pablo Picasso was stolen in 2010. The thief got scared and disposed it to trash shortly after the theft. The garbage was emptied and taken away, the painting lost for ever. Or the thief had been clever ...*

b) (3 points) Given a function $f(x,y)$ and a curve $\vec{r}(t)$. Let $L$ be the linearization of $f$ at $\vec{r}(0)$. Each of the following 3 vectors $\vec{a}, \vec{b}, \vec{c}$ is placed exactly twice in the puzzle to the right below.

| | |
|---|---|
| $\vec{a} = \nabla f(\vec{r}(0))$ | $\frac{d}{dt} f(\vec{r}(t))_{t=0} =$ [ ] $\cdot$ [ ] |
| $\vec{b} = \vec{r}\,'(0)$ | $L(\vec{r}(t)) =$ $f(\vec{r}(0)) +$ [ ] $\cdot$ [ ] |
| $\vec{c} = \vec{r}(t) - \vec{r}(0)$ | [ ] $= \lim_{t \to 0} \frac{1}{t}$ [ ] |

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the **skate board ramp**

$$z - f(x,y) = z - \sqrt{x^{30}y^3 + x} = 0$$

at the point $(1,2,3)$.

b) (5 points) Estimate

$$f(1.006, 1.98) = \sqrt{1.006^{30} \cdot 1.98^3 + 1.006}$$

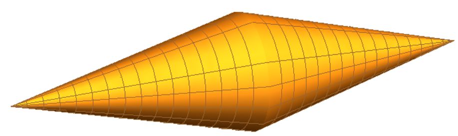
by linearizing the function $f(x,y)$ at $(1,2)$.

Problem 5) (10 points)

A **croissant** of length $2h$ and radius $r$ in the shape of two cones has fixed volume

$$V(r,h) = \frac{2\pi r^2 h}{3} = 18 \ .$$

Use Lagrange to find the values $r$ and $h$ for which the surface area

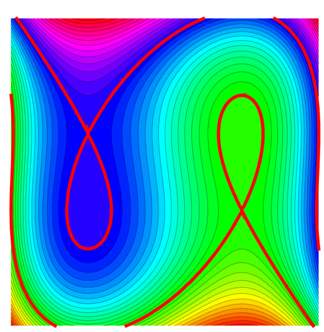


$$A(r,h) = 2\pi r\sqrt{r^2 + h^2}$$

is minimal. **Hint:** as you have seen in homework, it is much more convenient to minimize $f(r,h) = A(r,h)^2$ instead.

Problem 6) (10 points)

Find the local maxima, minima and saddle points of the **tadpole function**



$$f(x,y) = 3y^2 + 4x^3 + 2y^3 - 12x \ .$$

*First tadpole: "What is your favorite book?" Second tadpole: "Metamorphosis by Kafka. What is your favorite year?" First tadpole: "Leap year!" Both croak with laughter. Then, sick of frog jokes, they turn green.*

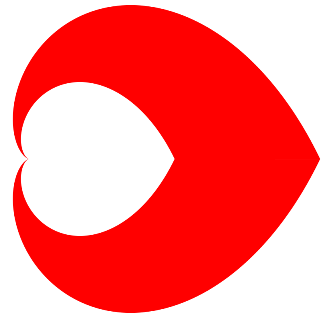
## Problem 7) (10 points)

Find the area of the heart shaped polar region

$$(\theta-\pi)^2 \leq r \leq 2(\theta-\pi)^2$$

with $0 \leq \theta \leq 2\pi$.

*Warning: Valentine cards displaying*
**"You are my** $r < (\theta-\pi)^2$**!"** *do not always work.*



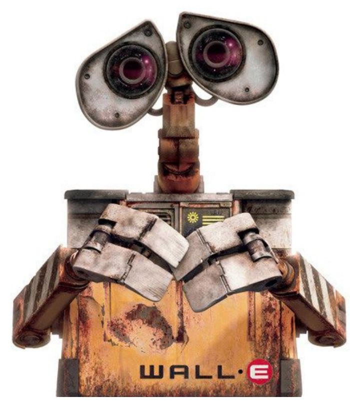
## Problem 8) (10 points)

Mathematica 10 does not give an elementary expression for the integral

$$\int_0^1 \int_{\exp(y)}^e \frac{1}{\log(x)}\, dx\, dy \ ,$$

where log is the natural log. You can! "Humans are awesome 2014"! https://www.youtube.com/watch?v=ZBCOMG2F2Zk

The logarithmic integral $\mathrm{Li}(x) = \int_0^x dt/\log(t)$ is important in number theory. It was Gauss who proposed first that the number $\pi(x)$ of primes smaller than $x$ is about $\mathrm{Li}(x)$. It is now known that $0.89\ \mathrm{Li}(x) \leq \pi(x) \leq 1.11\ \mathrm{Li}(x)$ for all large enough $x$. P.S. Mathematica can solve the double integral of course, but only if told to "FullSimplify".

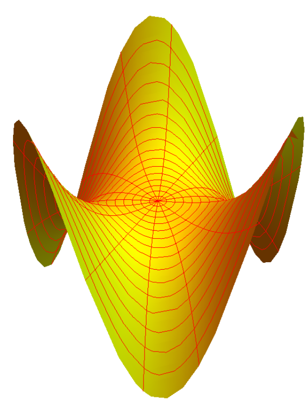



## Problem 9) (10 points)

Compute the weighted surface area

$$\int\int_R (u^2+v^2)|\vec{r}_u \times \vec{r}_v|\ dudv$$

of the monkey saddle parametrized by $\vec{r}(u,v) = [u, v, u^3 - 3uv^2]$ over the domain $R : u^2+v^2 \leq 1$. This quantity is also known as the moment of inertia of the surface. Spin that monkey!



Problem 10) (10 points)


The following two integrals are called "Mad Max" integrals because they were written while watching that movie:

a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)} \, dxdy \, .$$

b) (5 points) Integrate the double integral

$$\int \int_R \sin(x^2 + y^2) \, dxdy$$

where $R$ is the disk of radius $\sqrt{\pi/2}$.