11/5/2019 Second Hourly **Math 21a, Fall 2019**

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No functions were harmed in the making of this exam.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) T F

The chain rule assures that $\frac{d}{dt} f(\vec{r}(t))$ is the dot product of the gradient of f at $\vec{r}(t)$ and the velocity of $\vec{r}(t)$.

Solution:
This is a fact.

2) T F

If $f(x, y) = c$ is a curve defining $y = y(x)$ near $(x, y) = (1, 1)$ then $y'(1) = -f_x(1, 1)/f_y(1, 1)$.

Solution:
This is the implicit differentiation formula.

3) T F

For $\vec{v} = \nabla f / |\nabla f(1, 1, 1)|$, the directional derivative $D_{\vec{v}} f(1, 1, 1)$ is a vector normal to the level surface of f passing through $(1, 1, 1)$.

Solution:
The directional derivative is a scalar

4) T F

There exists a function $f(x, y)$ without any critical points.

Solution:
True. Every non-constant linear function for example.

5) T F

The function $f(x, y) = 6x + 17x^2 - 3y^2$ has a global maximum under the constraint $x = 0$.

Solution:
Just put $x = 0$ and see $-3y^2$.

6) T F

If $f(x, y)$ is a function satisfying $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) > 0$, then f has a minimum at $(0, 0)$.

Solution:
You can imagine something like $f(x, y) = xy + x^2/1000 + y^2/1000$ which is a saddle point.

7) T F The point $(1, 0)$ is the only critical point of $f(x, y) = xy - y$.

Solution:
Look at the gradient. It is $[y, x - 1]$ which gives $x = 1, y = 0$.

8) T F If $\vec{r}(u, v)$ parametrizes a level surface S then $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface $\vec{r}(u, v)$ for all (u, v).

Solution:
It is perpendicular to $\vec{r}_u$ or $\vec{r}_v$.

9) T F The surface area of a surface parametrized by $\vec{r}(u, v)$ over a domain R in the uv-plane is given by the integral $\int \int_R |\vec{r}_u \cdot \vec{r}_v| \, dudv$.

Solution:
We need $|\vec{r}_u \times \vec{r}_v|$.

10) T F If $f_{xy}(x, y) = 0$ for all (x, y) in the plane, then all critical points of f are saddle points.

Solution:
Take for example $x^2/2 + y^2/2 + xy = (x + y)^2/2$

11) T F If $(0, 0)$ is a saddle point for $f(x, y)$ with discriminant $D < 0$, then the directional derivative $D_{\vec{v}} f(0, 0)$ can take positive and negative values as $\vec{v}$ ranges over all unit vectors.

Solution:
All directional derivatives are zero.

12) T F If $(0, 0)$ is a saddle point for f, then $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ can have the same signs.

Solution:
Take $x^2 + y^2 + 100xy$ for example. It has a saddle point at $(0, 0)$.

13) T F

For $\vec{u} = [1,0]$ and $\vec{v} = [0,1]$ the discriminant D appearing in the second derivative test satisfies $D = (D_{\vec{u}}(D_{\vec{u}}f))(D_{\vec{v}}(D_{\vec{v}}f)) - (D_{\vec{u}}D_{\vec{v}}f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.

Solution:
This is the definition, if noticing that $f_u = D_u f$.

14) T F

If $\vec{r}(t)$ is a curve on the surface $f(x,y,z) = 1$, then the velocity vector $\vec{r}\,'(t)$ is perpendicular to $\nabla f(\vec{r}(t))$ for all t.

Solution:
Take the grand circle on the sphere. The normal vector is now perpendicular to the surface.

15) T F

The function $f(x,y) = 4x^3y^2 + \sin(\sin(x^{10})) + \cos(y^8)$ solves the partial differential equation $f_{xyxyxy} = 0$.

Solution:
Use Clairaut's theorem.

16) T F

If $f_x(x,y) = f_y(x,y)$ for all x, y, then $f_{xx}(x,y) = f_{yy}(x,y)$ for all (x,y).

Solution:
Differentiate f_x with respect to y to get f_{xy}. Differentiate f_y with respect to x to get f_{yx}. By Clairaut these are the same.

17) T F

The linearization of $f(x,y) = x^2+y^2$ at $(1,2)$ is $L(x,y) = 2x(x-1)+2y(y-2)$.

Solution:
The linearization is a linear function

18) T F

If $f(x,t)$ solves the partial differential equation $f_t = f_{xx}$, then $g(x,t) = f(x,-t)$ solves the partial differential equation $g_{tt} = g_{xx}$.

Solution:
You differentiate twice

19) | T | F | There is a function f such that $D_{\vec{u}} f(0,0)$ is positive for all unit vectors $\vec{u}$.

Solution:
If the function is positive in one direction, then it is negative in the opposite direction.

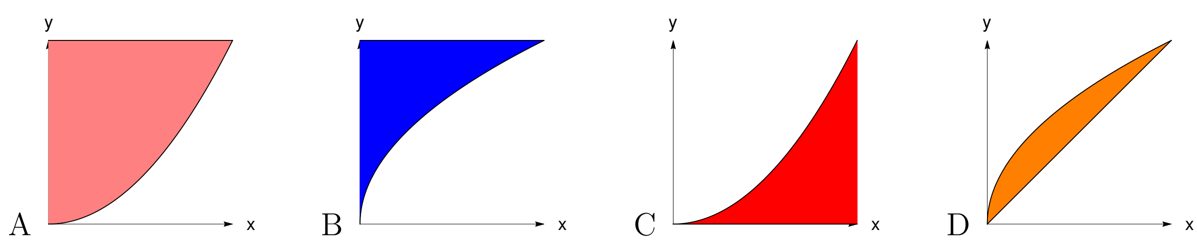
20) | T | F | The surface area of $z = x^2 + y^2$ with $x^2 + y^2 \leq 1$ does not depend on the parametrization.

Solution:
It is a general fact for parametrizations

Solution:
To summarize:
TTF TTF TTF FFT TTTT FFFT.
Which has a nice rhythm in it and also a positive aspect, as it is more true than false.

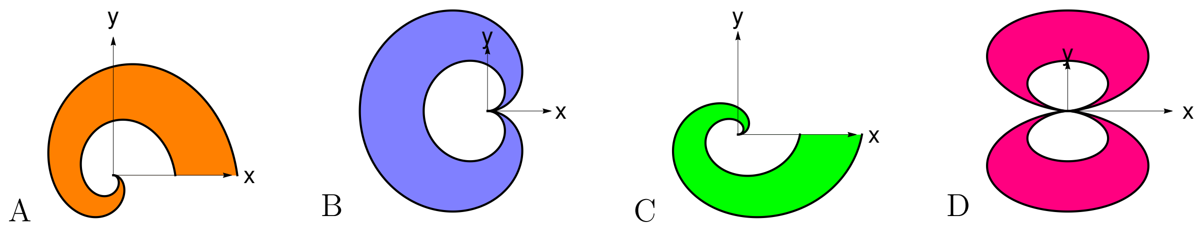
Problem 2) (10 points) No justifications are needed

a) (4 points) Match the following regions with their area computation. Each of the A-D will be used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_x^{\sqrt{x}} 1\, dydx$
	$\int_0^1 \int_{\sqrt{x}}^1 1\, dydx$
	$\int_0^1 \int_{\sqrt{y}}^1 1\, dxdy$
	$\int_0^1 \int_0^{\sqrt{y}} 1\, dxdy$

b) (4 points) Match the following regions with their area computation. Each of the A-D will be used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{\theta}^{2\theta} r \, dr d\theta$
	$\int_0^{2\pi} \int_{\pi-\theta/2}^{2\pi-\theta} r \, dr d\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r \, dr d\theta$
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r \, dr d\theta$

c) (2 points) Name all the four PDE's for a function $A(B, C)$ of variables B and C. Some equations come here with a constant parameter ν.

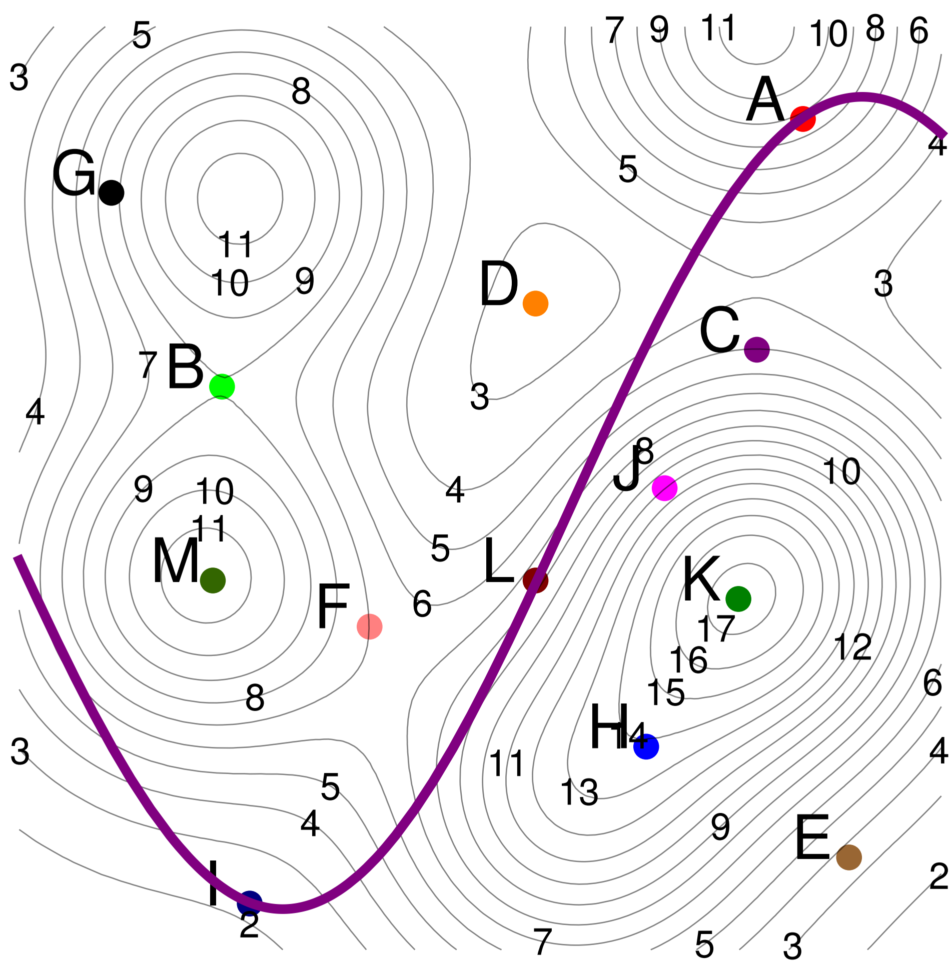
$A_{BB} + A_{CC} = 0$	$A_B + AA_C = \nu A_{CC}$	$A_{BB} = A_{CC}$	$A_B = \nu A_{CC}$

Solution:
a) DBCA
b) CABD
c) Laplace, Burgers, Wave, Heat.

Problem 3) (10 points) (No justifications are needed.)

(10 points) In the diagram below, the thin curves are the level curves = contours of an unknown smooth function, $f(x, y)$. The thick line is the constraint curve, $g(x, y) = y - \sin(x) = 0$. For each of the descriptions below, write clearly in the blank the label of the point in the diagram that best matches the description. You should use each of the labels A-M only once and leave out the three labels which do not match.

	Enter A-M
A local but not global maximum of $f(x,y)$.	
A global maximum of $f(x,y)$.	
The point among A-M with maximal $\|\nabla f\|$.	
A saddle point of f.	
A local minimum of $f(x,y)$.	
A local maximum of $f(x,y)$ on $\{g(x,y)=0\}$.	
A local minimum of $f(x,y)$ on $\{g(x,y)=0\}$.	
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	



Solution:
M,K,J,B,D,A,I,G,H,C.
Not used: L,E,G

Problem 4) (10 points)

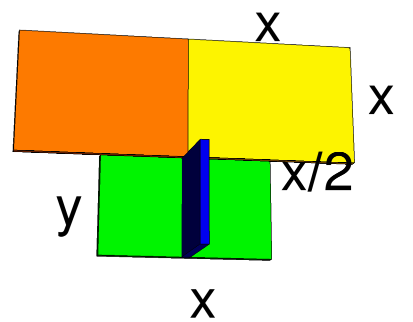
a) (5 points) The area function

$$f(x,y) = 2x^2 + xy + y^2$$

of an airplane has exactly one minimum under under the constraint

$$g(x,y) = 6x + 5y = 32 \ .$$

(You don't need to check this.) Find the minimum (x,y) using the method of Lagrange multipliers.



b) (5 points) Use the method of Lagrange multipliers to find (x,y) which maximizes the contour length $g(x,y) = 6x + 5y$ under the constraint $f(x,y) = 32$.

Solution:

a) The Lagrange equations

$$4x + y = \lambda 6, x + 2y = \lambda 5, 6x + 5y = 32$$

give one critical point $\boxed{x = 2, y = 4}$.

b) The Lagrange equations give now two critical points $x = 2, y = 4$ and $x = -2, y = -4$. The later point is not physical and also is a minimum and not a maximum of g. The solution is again $\boxed{x = 2, y = 4}$.

One can learn a bit here about a duality which matters in economics. Minimizing one quantity under the constraint of a second gives a maximum of the other quantity under the constraint of the first. However it is not quite equivalent. There was a minimum in b) which is not a maximum in a)!

Problem 5) (10 points)

Define the **duck function**

$$f(x,y) = 4x^2 - y^2 - 2x^4 + 2y \ .$$

a) (8 points) Find local maxima, local minima and saddle points of f using the second derivative test.

b) (1 point) Does $f(x,y)$ have a global maximum on the entire plane? No justification is needed.

c) (1 point) Does $f(x,y)$ have a global minimum on the entire plane? No justification is needed.

Any resemblance in this figure to actual persons or functions, living or dead is entirely coincidental.

Solution:
There are three critical points. The second derivative test gives

x	t	D	f_{xx}	nature
-1	1	32	-16	maximum
0	1	-16	8	saddle
1	1	32	-16	maximum

b) There is a global maximum (actually 2).

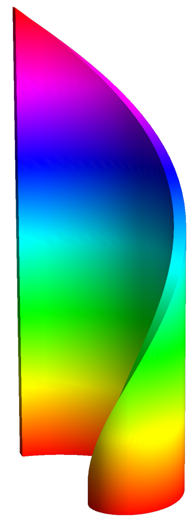
c) There is no global minimum.
The duck function can go arbitrary low. There is no limit, how low the duck function can go. One would have to put a constraint on it to get a minimum. Apropos similarities to actual things: the level curves of the function were seen in the duck figure, despite the legal disclaimer note.

Problem 6) (10 points)

a) (7 points) Set up the integral for the surface area of

$$\vec{r}(u,v) = \left[\ u\cos(u), u\sin(u), uv\ \right]$$

with $0 \leq v \leq 3$ and $0 \leq u \leq 4$. Simplify your answer as much as you can.



b) (3 points) Evaluate the integral.

Solution:
a) Compute $\vec{r}_u = [0,0,u], \vec{r}_v = [-u\sin(u)+\cos(u), u\cos(u)+\sin(u), 0]$, take the cross product to get $\vec{r}_u \times \vec{r}_v = [-u(u\cos(u)+\sin(u)), u(-u\sin(u)+\cos(u)), 0]$ which has length $|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2(u^2+1)} = u\sqrt{u^2+1}$. The integral is

$$\int_0^3 \int_0^4 u\sqrt{u^2+1}\, dudv\ .$$

b) We use substitution and get $3\int_0^4 u\sqrt{u^2+1}\, du = 3(u^2+1)^{3/2}(2/3) = 17^{3/2} - 1$.

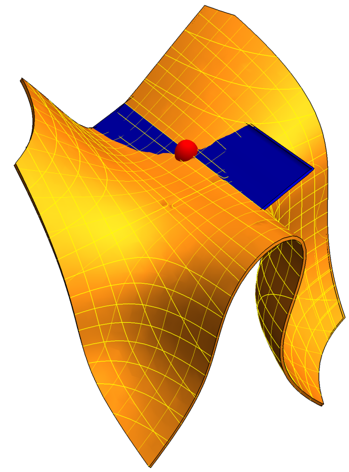
Problem 7) (10 points)

Knowing tangent planes can be important in **ray tracing**, where the computer bounces light around in the scene and reflects light at surfaces.

a) (6 points) Find the tangent plane to the surface

$$7xy^2z - x^2y + 2z^3 = 8$$

at the point (1, 1, 1).



b) (4 points) Near $(x, y, z) = (1, 1, 1)$, the surface can be written as $z = z(x, y)$. Find the gradient $[z_x(1, 1), z_y(1, 1)]$ at $(1, 1)$.

Solution:
a) $\nabla f = [7y^2z - 2xy, 14xyz - x^2, 7xy^2 + 6z^2]$ $\nabla f(1, 1, 1) = [5, 13, 13]$.
The equation is $\boxed{5x + 13y + 13z = 31}$.
b) $g_x = -f_x/f_z = -5/13, g_y = -f_y/f_z = -13/13 = -1$. So $\nabla g = [-5/13, -1]$.

Problem 8) (10 points)

We have a mystery function f which satisfies $f(0, 0) = 21$. We also know the two directional derivatives

$$D_{\vec{v}}f(0, 0) = \frac{10}{5}, \quad D_{\vec{w}}f(0, 0) = \frac{11}{5},$$

for the unit vectors $\vec{v} = [3, 4]/5$ and $\vec{w} = [4, 3]/5$.

a) (5 points) Find the tangent line to the curve $\{f(x, y) = 21\}$ at $(0, 0)$.

b) (5 points) Estimate $f(-0.01, 0.03)$.

Solution:

a) Write $\nabla f(0,0) = [a,b]$. The assumptions tell

$$[a,b] \cdot [3,4]/5 = 10/5, [a,b] \cdot [4,3]/5 = 11/5$$

This is a system of equations for a, b which has the solution $a = 2, b = 1$. The equation of the tangent plane is $ax + by = d$, where d is obtained by plugging in the point $(1,1)$. We have then $2x + y = 0$.

b) To estimate f, we use the linearization

$$L(-0.01, 0.03) = f(0,0) + a(-0.01 - 0) + b(0.03 - 0) = 21 + 2(-0.01) + 1 * 0.03 = 21.01 .$$

We did not know the function f but we could still use the directional derivative information to estimate the value of f near the point $(0,0)$.

Problem 9) (10 points)

Compute the following double integrals:

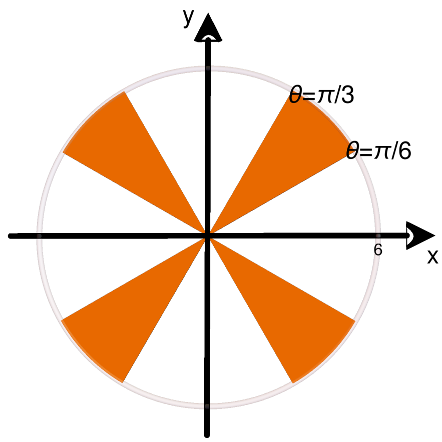
a) (5 points)

$$\iint_G x^2 + y^2 \; dxdy ,$$

where G is the propeller region given by

$$\{x^2 + y^2 \leq 36, y^2/3 < x^2 < 3y^2\} .$$

The propeller region consists of four blades, as you see in the picture.



b) (5 points)

$$\int_1^2 \int_0^{\sqrt{2-y}} \frac{\sin(\pi x)}{1 - x^2} \; dxdy .$$

Solution:

a)

$$4 \int_{\pi/6}^{\pi/3} \int_0^6 r^3 \; dr d\theta = 6^3 \pi = 216\pi$$

b) Switch the order of integration. To do so, make a picture.

$$\int_0^1 \int_1^{2-x^2} \frac{\sin(\pi x)}{1 - x^2} \; dydx .$$

Evaluating the inner integral gives $\frac{\sin(\pi x)}{1-x^2}[(2 - x^2 - 1)] = \sin(\pi x)$. Integrating this from 0 to π gives $\frac{2}{\pi}$.