THE AREA OF THE MANDELBROT SET

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Double integrals

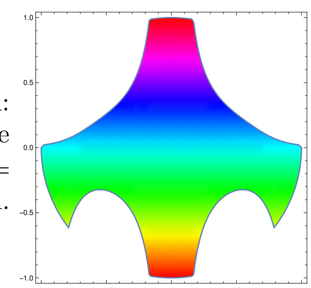
The **Riemann integral** is a limit of a summation process. If $f(x,y)$ is continuous on a region G, the integral $\iint_G f(x,y)\, dA$ is defined as the limit of the **Riemann sums**

$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n}) \in G} f(\frac{i}{n}, \frac{j}{n})$$

as $n \to \infty$. We write also $\iint_G f(x,y)\, dA$, where dA is a notation standing for "area element".

An example

Here is an example, where it is not difficult to give an answer. Question: Using a double integral, compute the area of the region between the curve $y = x^2$ and $y = x$, where $0 \leq x \leq 1$. Answer: The result is $\int_0^1 x - x^2\, dx = 1/2 - 1/3 = 1/6$. We can also compute it with a Monte Carlo computation.



```
n=10^6;N[Sum[{x,y}={Random[],Random[]}; If[x^2<y<x,1,0],{n}]/n]
```

In some cases, it is not possible to find an analytic expression.

$$G = \{(x,y) \mid x^6 y + \sin(y)^2 x^2 \cos(x^2) < 0.02, x^2 + y^2 < 1\}$$

would be difficult to describe analytically as we can not find expressions for the boundary. What still can find random points (x,y) in $[a,b] \times [c,d]$, then form

$$\frac{A}{n} \sum_{k=1}^{n} f(x_k, y_k)\ ,$$

where $A = (d-c)(b-a)$ is the area in which we shoot randomly.

```
n=10^6; S=Sum[{x,y}={2Random[]-1,2Random[]-1}; (4/n)*
   If[x^6 y+Sin[y]^2 x^2 Cos[x^2]< 0.02 && x^2+y^2<1,1,0],{n}]
```

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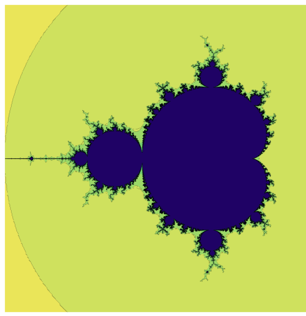
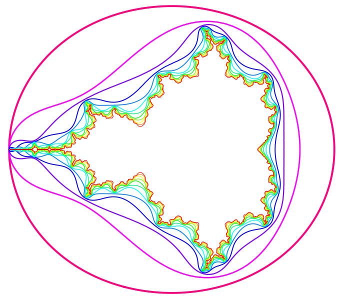
The area of the Mandelbrot set

The **Mandelbrot set** is

$$M = \{c = a + ib \in \mathbb{C} \in \mathbb{R}^2 \mid T_c^n(0) \text{ stays bounded } \},$$

where $T_c(z) = z^2 + c$. In real coordinates the map is $T_c(x, y) = (x^2 - y^2 + a, 2xy + b)$. The notation T^n means applying T a number of times. For example $T^3(x, y) = T(T(T(x, y)))$. One can draw it numerically or approximate curves one knows to be outside.





What is the area of M? We know it is contained in the rectangle $x \in [-2, 1]$ and $y \in [-3/2, 3/2]$ which has area 9. One way to get bounds on the area is to approximate the set with polynomials p_n. Here is the code to plot such polynomial **contour curves** and to compute its area (which is already challenging for a computer algebra system for n=2.

```
p[0_,z_]:= z; p[n_,z_]:=p[n-1,z]^2+z;
ContourPlot[Abs[p[8,x+I y]]==2,{x,-2,1}, {y,-3/2,3/2}]
f=p[2,x+I*y]*p[2,x-I*y] // Expand
```

The polynomial $p_2(x, y)$ is

$$x^8+4x^7+4x^6y^2+6x^6+12x^5y^2+6x^5+6x^4y^4+14x^4y^2+5x^4+12x^3y^4+4x^3y^2+2x^3+4x^2y^6+10x^2y^4+2x^2y^2+x^2+4xy^6-2xy^4+2xy^2+y^8+2y^6-3y^4+y^2$$

But computing the area of such regions bounded by polynomial curves is already difficult for computer algebra systems

```
Area[ImplicitRegion[f<4,{x,y}]]
```

We can also just randomly shoot into this rectangle and see whether we are in the Mandelbrot set or not after 1000 iterations:

```
M=Compile[{x,y},Module[{z=x+I y,k=0},
  While[Abs[z]<2.&&k<999,z=N[z^2+x+I y];++k];Floor[k/999]]];
n=10^6; 9*Sum[M[-2+3*Random[],-1.5+3*Random[]],{n}]/n
```

Here is a value computed with this code:

$$1.51116$$

How accurate do we know the Mandelbrot set? It is a data problem. One particular source gives the bounds $[1.50311, 1.5613027]$ (a preprint of Y. Fisher and J. Hill, Bounding the Area of the Mandelbrot Set, Numerische Mathematik). See also **https://arxiv.org/pdf/1410.1212.pdf**.