**12/13/2019 FINAL EXAM** **Math 21a, Fall 2019**

Name:

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| MWF 9 Oliver Knill |
| MWF 9 Arnav Tripathy |
| MWF 9 Tina Torkaman |
| MWF 10:30 Jameel Al-Aidroos |
| MWF 10:30 Karl Winsor |
| MWF 10:30 Drew Zemke |
| MWF 12 Stepan Paul |
| MWF 12 Hunter Spink |
| MWF 12 Nathan Yang |
| MWF 1:30 Fabian Gundlach |
| MWF 1:30 Flor Orosz-Hunziker |
| MWF 3 Waqar Ali-Shah |

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and defined everywhere unless stated otherwise.
- **Show your work**. Except for problems 1-3 and 6, we need to see details of your computation. If you are using a theorem for example, state the theorem.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

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| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| Total: | | 140 |

| Problem 1) True/False questions (20 points). | No justifications are needed. |
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| | | | |
|---|---|---|---|
| 1) | T | F | The distance between a point $P$ and a line $L$ is zero if and only if the point is on the line. |
| 2) | T | F | The curves $\vec{r}(t) = [2t, 3t, 0]$ and $\vec{s}(t) = [3t, -2t, 1]$ intersect at a right angle at $(0, 0, 0)$. |
| 3) | T | F | The surface $z^2 - (x-3)^2 - (y+1)^2 = -1$ is a one-sheeted hyperboloid. |
| 4) | T | F | The area of a surface parametrized by $\vec{r}(u, v) = [u, g(u, v), v]$ is given by $\iint_R \lvert g_u \times g_v \rvert \; dudv$. |
| 5) | T | F | If $\vec{r}(0) = \vec{r}\,'(0) = \vec{0}$, then $\vec{r}(t)$ is zero for all times $t > 0$. |
| 6) | T | F | If $\vec{F} = \nabla f$ and $\vec{r}(t)$ is a curve, then $\frac{d}{dt} f(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$. |
| 7) | T | F | If $\vec{F}$ and $\vec{G}$ are vector fields in $\mathbf{R}^2$ for which the curl is constant 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field. |
| 8) | T | F | The solid $1 \leq x^2 + y^2 + z^2 \leq 2$ in $\mathbf{R}^3$ is simply connected. |
| 9) | T | F | The parametrization $\vec{r}(u, v) = [u^3, u^6 - v^6, v^3]$ describes a hyperbolic paraboloid (saddle surface). |
| 10) | T | F | The flux of the curl of the field $\vec{F}(x, y, z) = [x, y, z]$ through $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ oriented outwards is $4\pi$. |
| 11) | T | F | There exists a vector field $\vec{F}$ in space $\mathbf{R}^3$ such that $\text{curl}(\vec{F}) = [5x, -11y, 7z]$. |
| 12) | T | F | The flux of $\vec{F} = [x, 0, 0]$ through the outwards oriented surface bounding the cuboid $\lvert x \rvert \leq 1, \lvert y \rvert \leq 2, \lvert z \rvert \leq 3$ is equal to 24. |
| 13) | T | F | The points that satisfy $\theta = \pi/4$ and $\phi = \pi/4$ in spherical coordinates form a surface which is part of a cone. |
| 14) | T | F | If $f(x, y, z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere. |
| 15) | T | F | For any vector field $\vec{F}$ and any parametrized curve $\vec{r}(t)$ with $t \in [a, b]$ we have $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$. |
| 16) | T | F | There exists a vector field $\vec{F}$ and a scalar function $g(x, y, z)$ such that $\text{curl}(\vec{F}) = \text{grad}(g)$. |
| 17) | T | F | If $\vec{F}$ is a vector field in space and $S$ is a surface bounding a solid and the surface is oriented outwards, then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$. |
| 18) | T | F | The solid defined by $x^2 + y^2 + z^2 \leq 9, z \leq \sqrt{x^2 + y^2}$ has volume $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \; d\rho d\phi d\theta$. |
| 19) | T | F | For any vector field $\vec{F} = [P, Q, R]$, we have $\lvert \text{div}(\vec{F}) \rvert \leq \lvert \text{curl}(\vec{F}) \rvert$. |
| 20) | T | F | If $\int_a^b \int_c^d f(x, y) \; dxdy = \int_c^d \int_a^b f(x, y) dxdy$ for all $a, b, c, d$, then $f(x, y) = f(y, x)$ for all $x, y$. |

| Problem 2) (10 points) |
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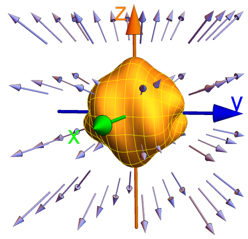
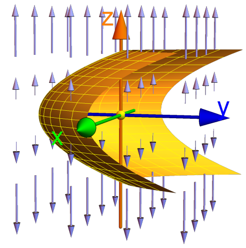
a) (2 points) We decide in two cases whether the **flux** of the vector field through the surface $S$ is positive or negative. In the left picture (belonging to the boxes to the left), $S$ is oriented outwards, in the right picture $S$ is oriented in the positive $y$ direction.

The flux through the left surface is

| |
|---|
| Positive |
| Negative |



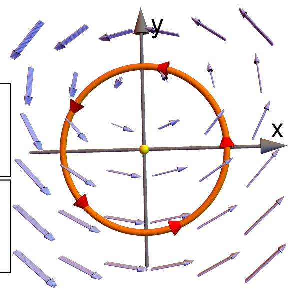

The flux through the right surface is

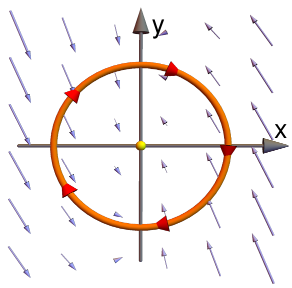
| |
|---|
| Positive |
| Negative |

b) (2 points) Decide in each case, whether the **line integral** of the vector field along the closed circular loop is positive or negative. Check the boxes on each side.

The line integral along the curve is

| |
|---|
| Positive |
| Negative |




The line integral along the curve is

| |
|---|
| Positive |
| Negative |

c) (2 points) Check on each side the box, for which the answer is positive $> 0$. Check exactly one box on the left for the left vector field and one box on the right for the right vector field.

For the field to the left: (check one box)

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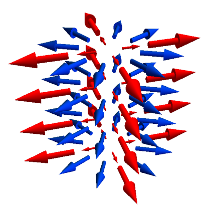
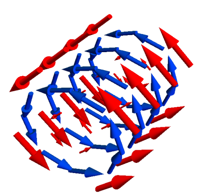
| $\|\text{curl}(\vec{F})\| > 0$ |
| $\text{div}(\vec{F}) > 0$ |





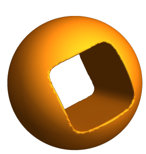
For the field to the right: (check one box)

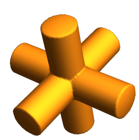
| |
|---|
| $\|\text{curl}(\vec{F})\| > 0$ |
| $\text{div}(\vec{F}) > 0$ |

d) (2 points) Decide for each of the solids whether it is simply connected.

The solid to the left is (check one box)

| |
|---|
| simply connected |
| not simply connected |





The solid to the right is (check one box)

| |
|---|
| simply connected |
| not simply connected |

e) (2 points) Write down the names of two partial differential equations involving functions $f(t, x)$ for which only the first derivative with respect to $t$ appear.

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| Problem 3) (10 points) | No justifications are needed. |
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In this problem we repeat a routine task practiced already in the homework. The problem deals with the **three fundamental derivative operations** div, curl and grad defined in multivariable calculus. There are 27 ways to combine three such operations and being interested in combinatorics, we wonder how many operations are defined. To find out, we select in problems a) to c) the cases which are NOT defined.

a) (3 points) In the following table, cross out every expression which is NOT defined. Let $f(x,y,z)$ be a function of three variables.

| | | |
|---|---|---|
| grad(grad(grad($f$))) | grad(curl(grad($f$))) | grad(div(grad($f$))) |
| curl(grad(grad($f$))) | curl(curl(grad($f$))) | curl(div(grad($f$))) |
| div(grad(grad($f$))) | div(curl(grad($f$))) | div(div(grad($f$))) |

b) (3 points) In the following table, cross out every expression which is NOT defined. Let $\vec{F} = [P,Q,R]$ denote a vector field in $\mathbf{R}^3$.

| | | |
|---|---|---|
| grad(grad(curl($\vec{F}$))) | grad(curl(curl($\vec{F}$))) | grad(div(curl($\vec{F}$))) |
| curl(grad(curl($\vec{F}$))) | curl(curl(curl($\vec{F}$))) | curl(div(curl($\vec{F}$))) |
| div(grad(curl($\vec{F}$))) | div(curl(curl($\vec{F}$))) | div(div(curl($\vec{F}$))) |

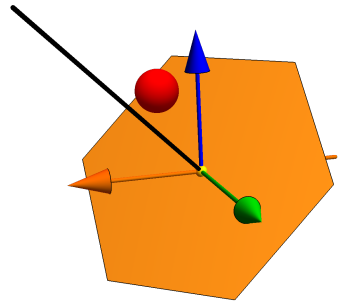
c) (3 points) In the following table, cross out every expression which is NOT defined. Again, $\vec{F} = [P,Q,R]$ is a vector field in $\mathbf{R}^3$.

| | | |
|---|---|---|
| grad(grad(div($\vec{F}$))) | grad(curl(div($\vec{F}$))) | grad(div(div($\vec{F}$))) |
| curl(grad(div($\vec{F}$))) | curl(curl(div($\vec{F}$))) | curl(div(div($\vec{F}$))) |
| div(grad(div($\vec{F}$))) | div(curl(div($\vec{F}$))) | div(div(div($\vec{F}$))) |

d) (1 point) Two of the following expressions are always zero (either the zero number or zero vector). Which ones? As before, $\vec{F} = [P,Q,R]$ is a vector field and $f(x,y,z)$ a scalar function in $\mathbf{R}^3$

| | |
|---|---|
| curl(grad($f$)) | curl(curl($\vec{F}$)) |
| div(grad($f$)) | div(curl(($\vec{F}$) |

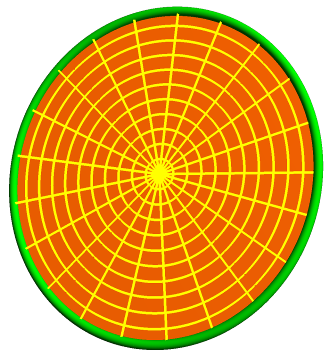
Problem 4) (10 points)



a) (5 points) Find the distance of the point $P = (3, 4, 5)$ to the line $\vec{r}(t) = [t, t, t]$.

b) (5 points) Find the distance of the point $P = (3, 4, 5)$ to the plane $x + y + z = 0$.

Problem 5) (10 points)



a) (5 points) Find the surface area of the **spider web** surface parametrized by

$$\vec{r}(u, v) = [v\sqrt{2}\cos(u), v\sin(u), v\sin(u)]$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

b) (5 points) Find the arc length of the boundary curve

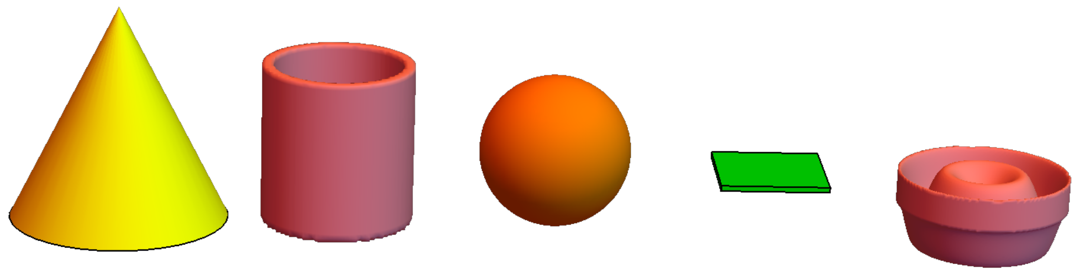
$$\vec{r}(t) = [\sqrt{2}\cos(t), \sin(t), \sin(t)]$$

with $0 \leq t \leq 2\pi$.

Problem 6) (10 points) No justifications are needed.

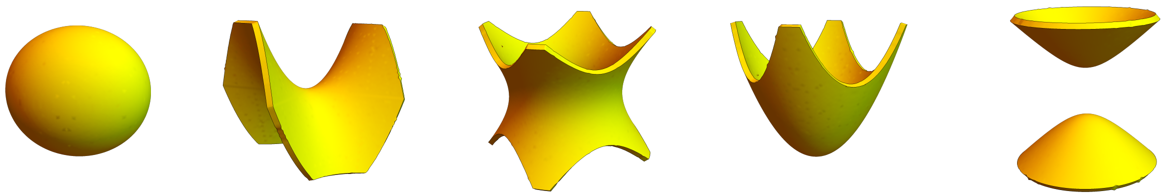
a) (5 points) We **teach an AI** to recognize objects and ask to match the pictures with the parametric surface. Do the job for the AI and match the parametrizations which best suit the surfaces. In each case, the parameters $(u, v)$ range over some region $R$ which are not given as it is irrelevant for the matching task here.

a) $\vec{r}(u, v) = [\sin(v)\cos(u), \sin(v)\sin(u), \cos(v)]$
b) $\vec{r}(u, v) = [u, v, \sin(u^2 + v^2)]$
c) $\vec{r}(u, v) = [(1 - u)\cos(v), (1 - u)\sin(v), u]$
d) $\vec{r}(u, v) = [\cos(v), \sin(v), (1 - u)]$
e) $\vec{r}(u, v) = [(1 - u), v, 1]$



Enter the letters a)-e) into the boxes below the surfaces. There is an exact match.

b) (5 points) A **CAPTCHA** is a test designed to distinguish humans from AI's. The task is to recognize and label objects. To show that you are human, name the surfaces in one or two words each.



| Problem 7) (10 points) |
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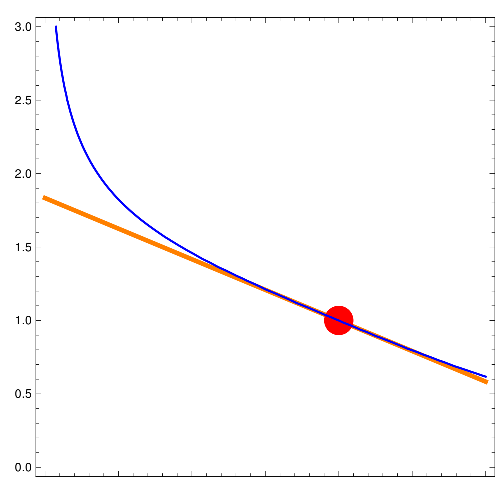
a) (2 points) Find the **tangent line** to the curve defined by

$$f(x,y) = x^2y + y^4x = 6$$

at the point $(2,1)$.
b) (2 points) Near $(2,1)$, the curve can be written as a **graph** $y = g(x)$. Find $g'(2)$.
c) (2 points) Find $L(x,y)$, the **linearization** of $f$ at the point $(2,1)$.
d) (2 points) **Estimate** $f(2.001, 1.0001)$ using linearization.
e) (2 points) What is the **directional derivative** $D_{[3,4]/5}f(2,1)$?



Problem 8) (10 points)

a) (7 points) Classify all the critical points of the **1001 nights function**

$$f(x, y) = x^{1001} - 1001x + y^{1001} - 1001y$$

using the second derivative test.

b) (2 points, please justify briefly) Is there a global maximum of $f$ on $x^2 + y^2 \leq 1$?

c) (1 point, no justification needed) Is there a global maximum of $f$ on $\mathbf{R^2}$?

Problem 9) (10 points)

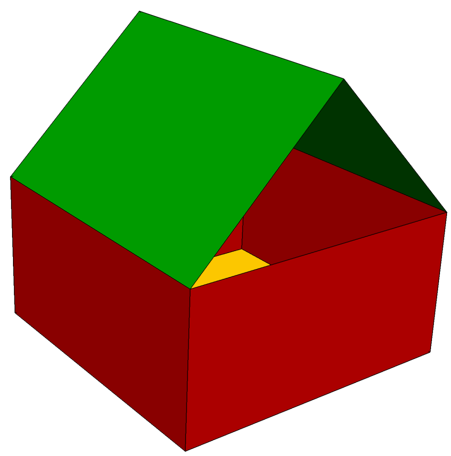
We want to minimize the paper used to build a **card house** of a given volume with 4 cards of size $x \times y$ and 3 cards of size $x \times x$. The area function

$$f(x, y) = 4xy + 3x^2$$

has a global minimum under the condition that the volume

$$g(x, y) = x^2 y = 3/2$$

is fixed and $x > 0, y > 0$. Use Lagrange multipliers to find the minimal value $f(x, y)$.



Problem 10) (10 points)

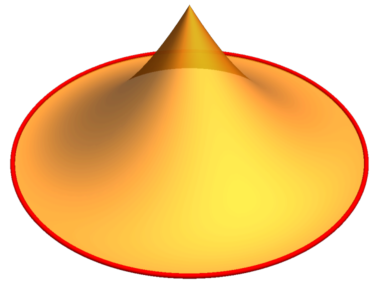
Find the flux of the curl of

$\vec{F}(x,y,z) = [-y(x+1)+z^7\cos(z), z, z^{21}+z\cos(xy)]$

through the surface $S$ parametrized by

$\vec{r}(t,s) = [s\cos(t), s\sin(t), (1-s)^2], 0 \leq t \leq 2\pi, 0 \leq s \leq 1$ .

The orientation of $S$ is **downwards**.



Problem 11) (10 points)

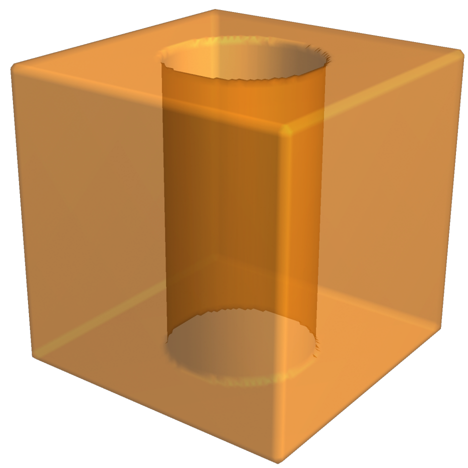
Find the flux $\iint_S \vec{F} \cdot d\vec{S}$ of the vector field

$\vec{F}(x,y,z) = [y^3+\sin(y), z^3+\sin(z), 5z+\sin(yx)]$

through the surface $S$ bounding the region

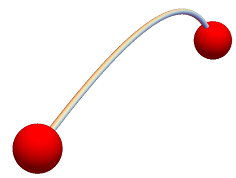
$$x^2 \leq 4, y^2 \leq 4, z^2 \leq 4, x^2+y^2 \geq 1 .$$

The surface $S$ is oriented outwards.



Problem 12) (10 points)

a) (4 points) Assuming $\vec{r}''(t) = [6, 0, 0]$ for all $t$ and $\vec{r}(0) = [0, 3, 4]$ and $\vec{r}'(0) = [0, 0, 1]$, find $\vec{r}(1)$.



b) (6 points) Using the curve $\vec{r}(t)$ from a) and **using an integral theorem**, compute the line integral of the vector field

$$\vec{F}(x, y, z) = [\pi \cos(\pi x), 3y^2 + z, 4z^3 + y]$$

along the path $\vec{r}(t)$ from $t = 0$ to $t = 1$.

| Problem 13) (10 points) |
|---|

a) (5 points) Find the volume of the solid which is given in **cylindrical coordinates** $(r, \theta, z)$ as

$$1 \leq r \leq 3 + \cos(8\theta), 1 \leq z \leq 4 \ .$$

b) (5 points) Find the area of the **starfleet logo** bounded by

$$\vec{r}(t) = [3\cos(t) + 3\sin(2t), 4\sin(t) + 3\sin(2t)]$$

with $0 \leq t \leq 2\pi$. The region is illustrated to the right.